Chapter 5

OFDM

Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

OFDM: Overview

- Let $S_1, S_2, \ldots, S_N$ be the information symbol.
- The discrete baseband OFDM modulated symbol can be expressed as

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k t}{T_s} \right), \quad 0 \le t \le T_s$$

$$= \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left(j \frac{2\pi k t}{T_s} \right)}_{c_k(t)}$$

Some references may use different constant in the front

Some references may start with different time interval, e.g. [-T$_s$/2, +T$_s$/2]

Note that:

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\mathrm{Re}\{S_k\} \cos\left(\frac{2\pi k t}{T_s} \right) - \mathrm{Im}\{S_k\} \sin\left(\frac{2\pi k t}{T_s} \right) \right)$$

OFDM Application

- 802.11 Wi-Fi: a and g versions
- DVB-T (the terrestrial digital TV broadcast system used in most of the world outside North America)
- DMT (the standard form of ADSL - Asymmetric Digital Subscriber Line)
- WiMAX

Wireless	Wireline
IEEE 802.11a, g, n (WiFi) Wireless LANs	ADSL and VDSL broadband access via POTS copper wiring
IEEE 802.15.3a Ultra Wideband (UWB) Wireless PAN	MoCA (Multi-media over Coax Alliance) home networking
IEEE 802.16d, e (WiMAX), WiBro, and HiperMAN Wireless MANs	PLC (Power Line Communication)
IEEE 802.20 Mobile Broadband Wireless Access (MBWA)	
DVB (Digital Video Broadcast) terrestrial TV systems: DVB-T, DVB-H, T-DMB, and ISDB-T	
DAB (Digital Audio Broadcast) systems: EUREKA 147, Digital Radio Mondiale, HD Radio, T-DMB, and ISDB-TSB	
Flash-OFDM cellular systems	
3GPP UMTS & 3GPP@ LTE (Long-Term Evolution) and 4G	

We shall focus on the single user case of OFDM.

Motivation

Why do ne need OFDM?

- First, we study the wireless channel.
- There are a couple of difficult problems in communication system over wireless channel.
- Also want to achieve high data rate (throughput)

Chapter 5
OFDM

5.1 Wireless Channel

Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

Single Carrier Transmission

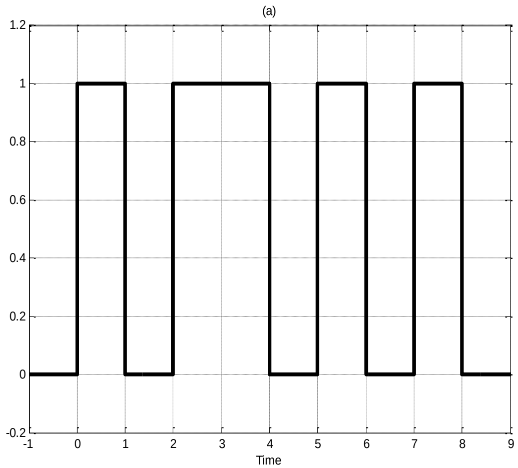
- Baseband:

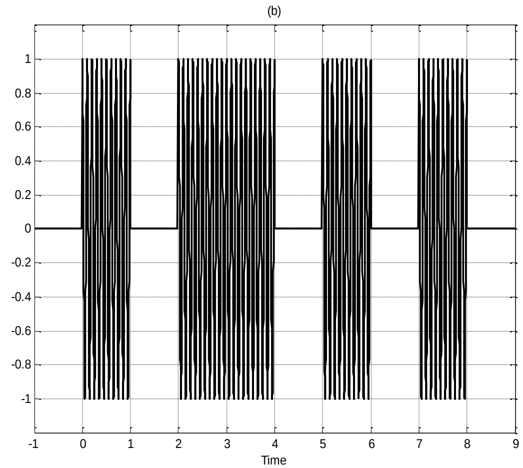
$$s(t) = \sum_{k=0}^{N-1} s_k p(t - kT_s)$$

$$p(t) = 1_{[0,T_s)}(t) = \begin{cases} 1, & t \in [0, T_s) \\ 0, & \text{otherwise.} \end{cases}$$

- Passband:

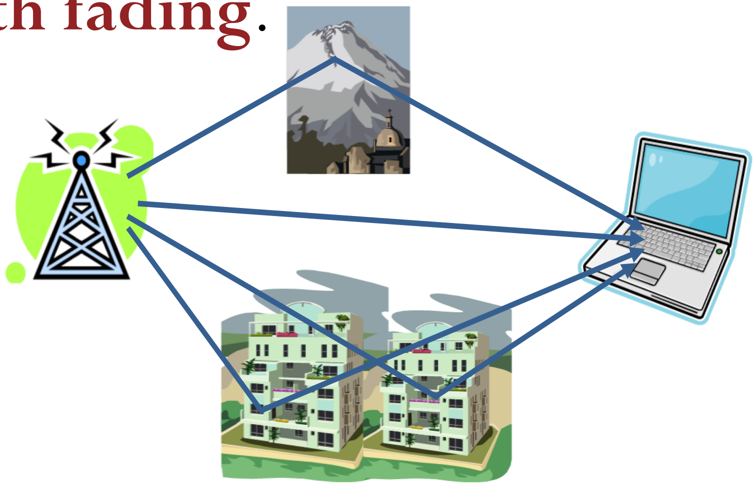
$$x(t) = \mathrm{Re}\left\{ s(t) e^{j2\pi f_c t} \right\}$$



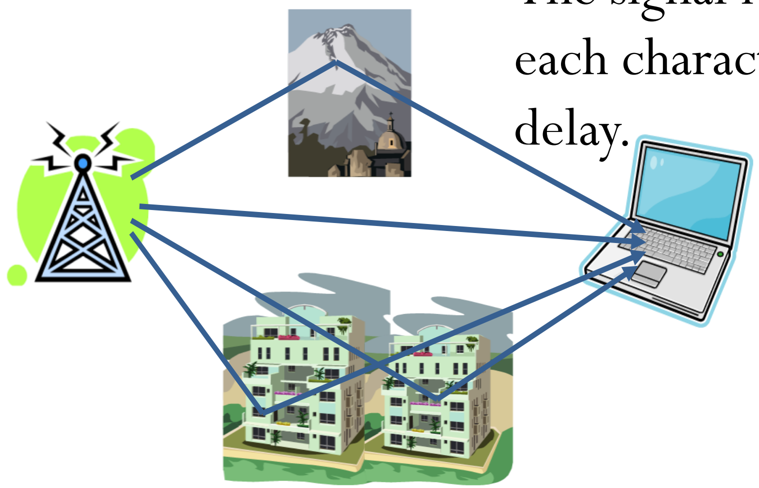


Multipath Propagation

- In a wireless mobile communication system, a transmitted signal propagating through the wireless channel often encounters multiple reflective paths until it reaches the receiver
- We refer to this phenomenon as **multipath propagation** and it causes fluctuation of the amplitude and phase of the received signal.
- We call this fluctuation **multipath fading**.



Wireless Comm. and Multipath Fading



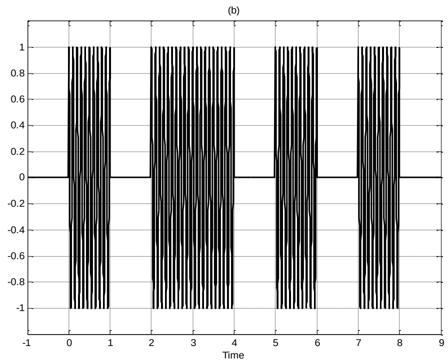
The signal received consists of a number of reflected rays, each characterized by a different amount of attenuation and delay.

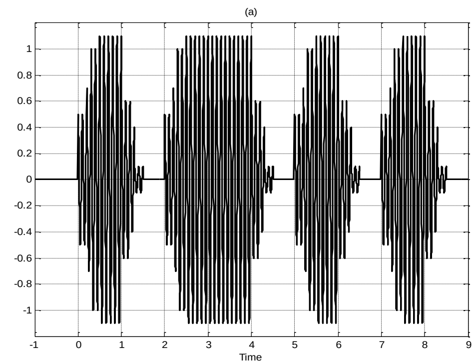
$$r(t) = x(t) * h(t) + n(t) = \sum_{i=0}^{v} \beta_i x(t-\tau_i) + n(t)$$

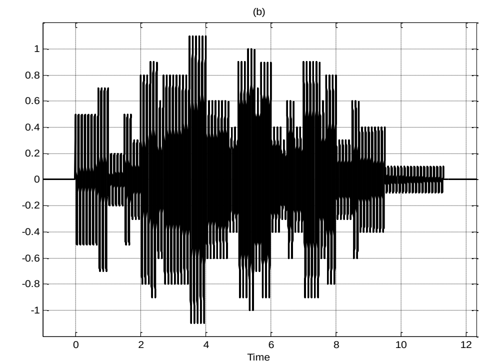
$$h(t) = \sum_{i=0}^{v} \beta_i \delta(t-\tau_i)$$

$$h_1(t) = 0.5\delta(t) + 0.2\delta(t-0.2T_s) + 0.3\delta(t-0.3T_s) + 0.1\delta(t-0.5T_s)$$

$$h_2(t) = 0.5\delta(t) + 0.2\delta(t-0.7T_s) + 0.3\delta(t-1.5T_s) + 0.1\delta(t-2.3T_s)$$

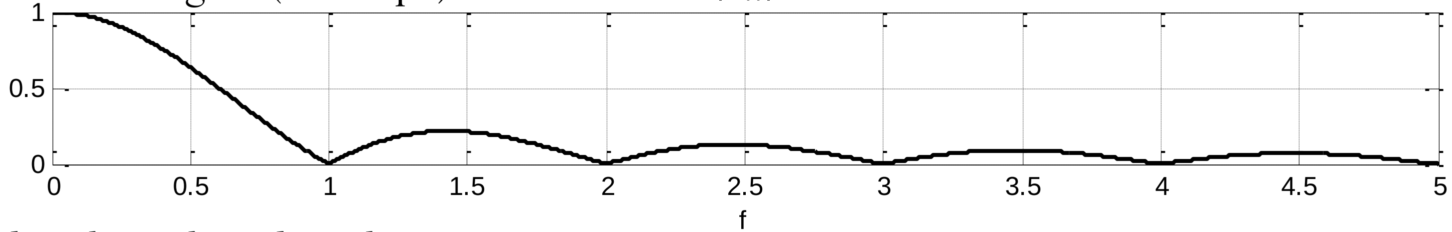




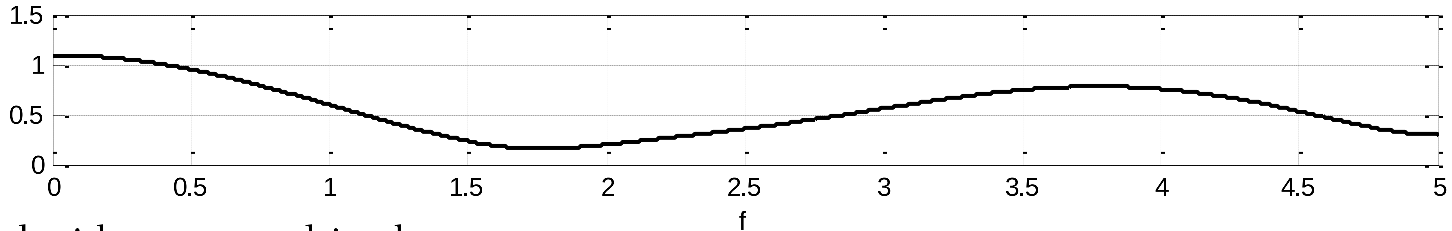


Frequency Domain

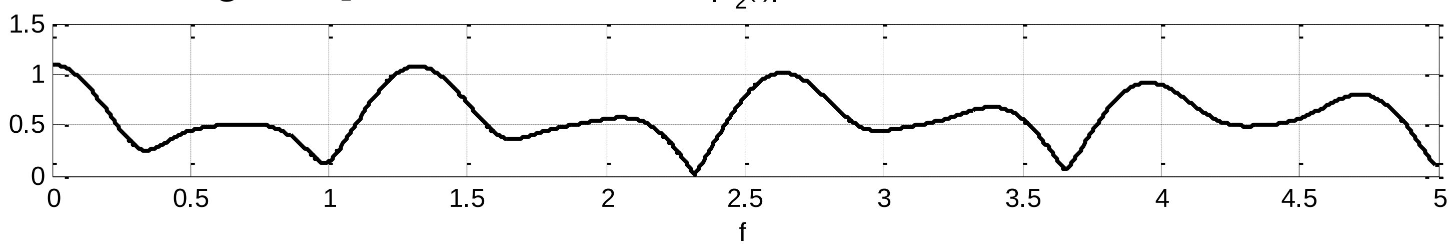
The transmitted signal (envelope)



Channel with weak multipath



Channel with strong multipath



COST 207 Channel Model

- Based on channel measurements with a bandwidth of 8–10MHz in the 900MHz band used for 2G systems such as GSM.

Path #	Rural Area (RA)		Typical Urban (TU)		Bad Urban (BU)		Hilly Terrain (HT)	
	Delay	Power	Delay	Power	Delay	Power	Delay	Power
	(µs)	(dB)	(µs)	(dB)	(µs)	(dB)	(µs)	(dB)
1	0	0	0	–3	0	–2.5	0	0
2	0.1	–4	0.2	0	0.3	0	0.1	–1.5
3	0.2	–8	0.5	–2	1.0	–3	0.3	–4.5
4	0.3	–12	1.6	–6	1.6	–5	0.5	–7.5
5	0.4	–16	2.3	–8	5.0	–2	15.0	–8.0
6	0.5	–20	5.0	–10	6.6	–4	17.2	–17.7

[Fazel and Kaiser, 2008, Table 1-1]

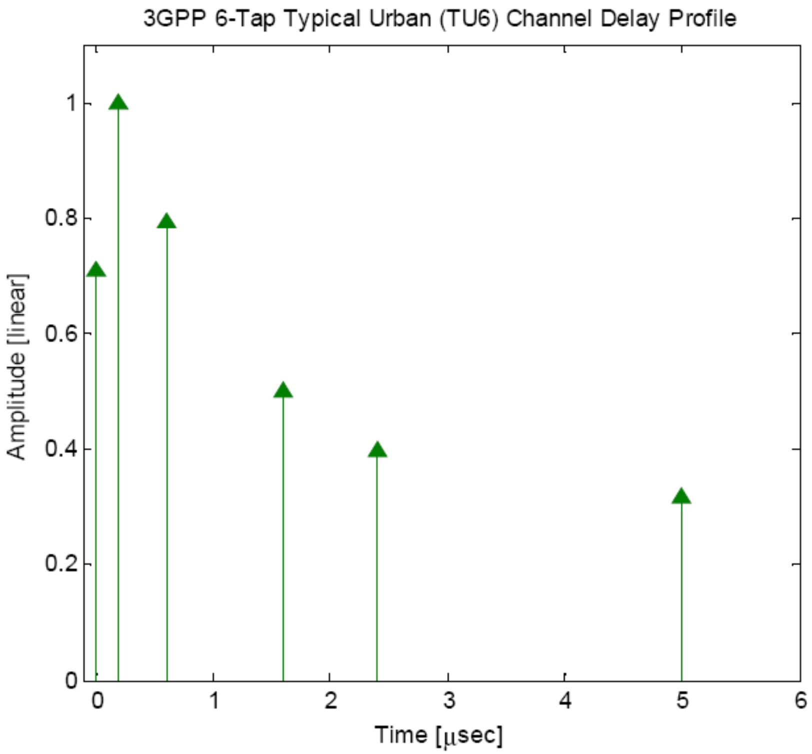
3GPP LTE Channel Modelss

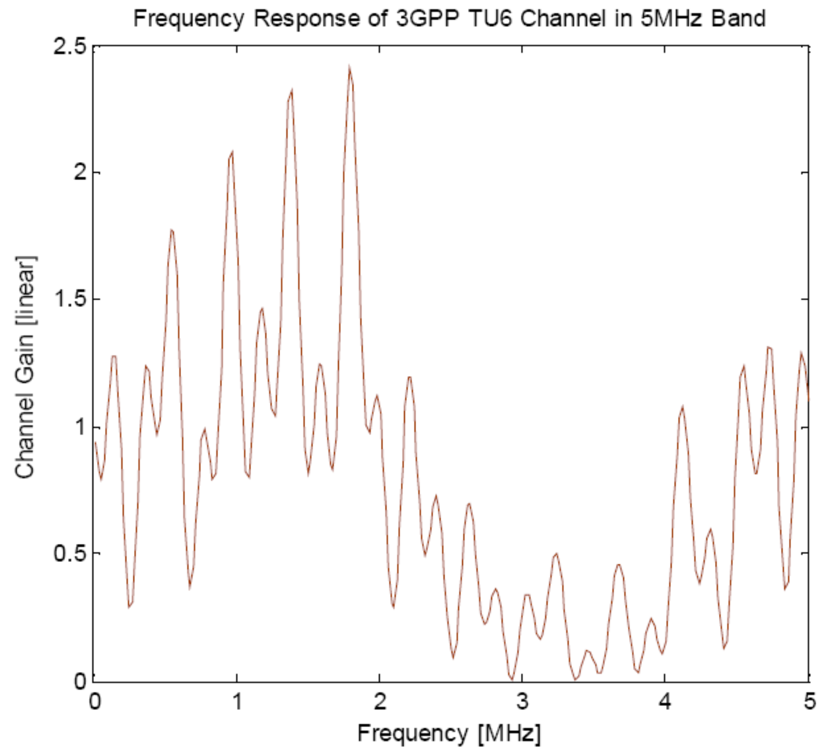
Path number	Extended Pedestrian A (EPA)		Extended Vehicular A (EVA)		Extended Typical Urban (ETU)	
	Delay (ns)	Power (dB)	Delay (ns)	Power (dB)	Delay (ns)	Power (dB)
1	0	0	0	0	0	−1
2	30	−1	30	−1.5	50	−1
3	70	−2	150	−1.4	120	−1
4	90	−3	310	−3.6	200	0
5	110	−8	370	−0.6	230	0
6	190	−17.2	710	−9.1	500	0
7	410	−20.8	1090	−7	1600	−3
8			1730	−12	2300	−5
9			2510	−16.9	5000	−7

[Fazel and Kaiser, 2008, Table 1-3]

3GPP 6-tap typical urban (TU6)

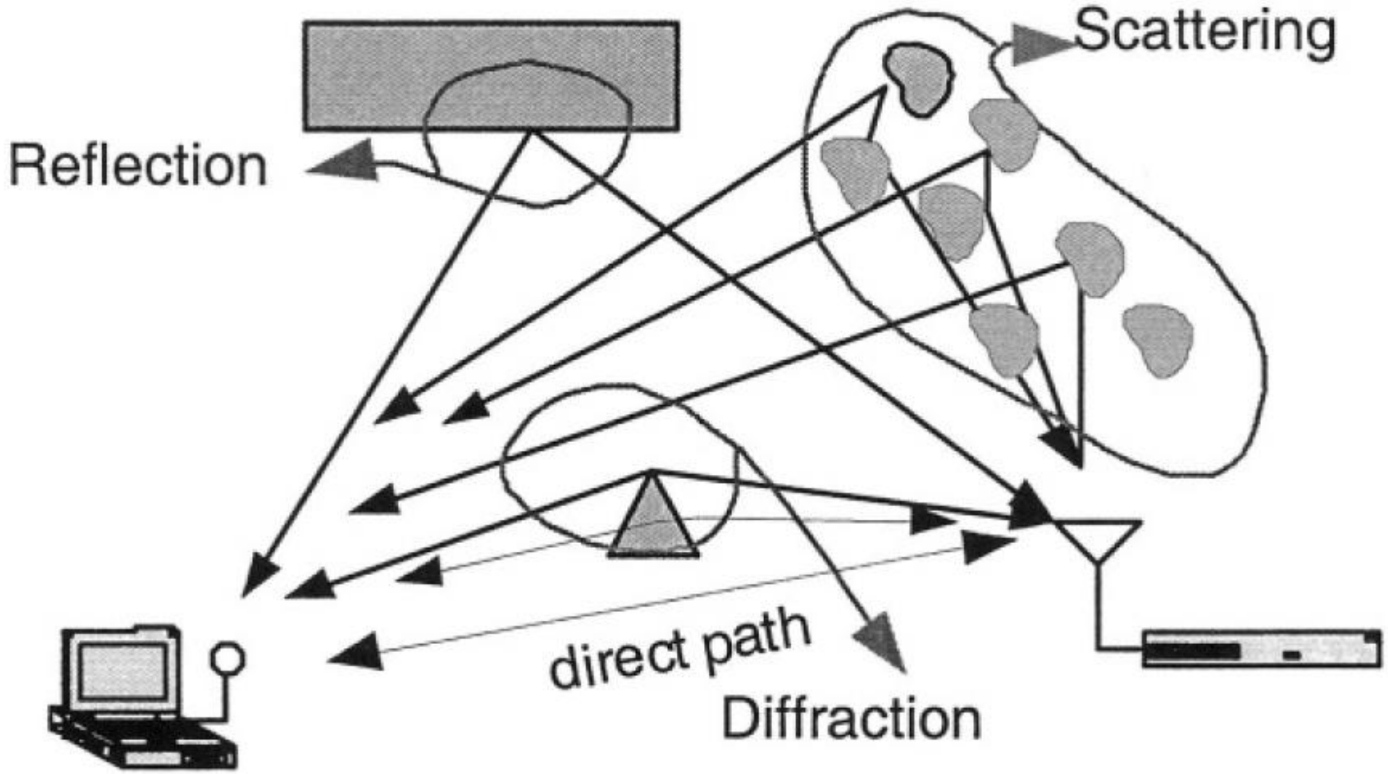
- Delay profile and frequency response of 3GPP 6-tap typical urban (TU6) Rayleigh fading channel in 5 MHz band.





[3GPP TS 45.005 – 3GPP; Technical Specification Group GSM/EDGE Radio Access Network; Radio Transmission and Reception (Release 7)]

Wireless Propagation



[Bahai, 2002, Fig. 2.1]

Three steps towards modern OFDM

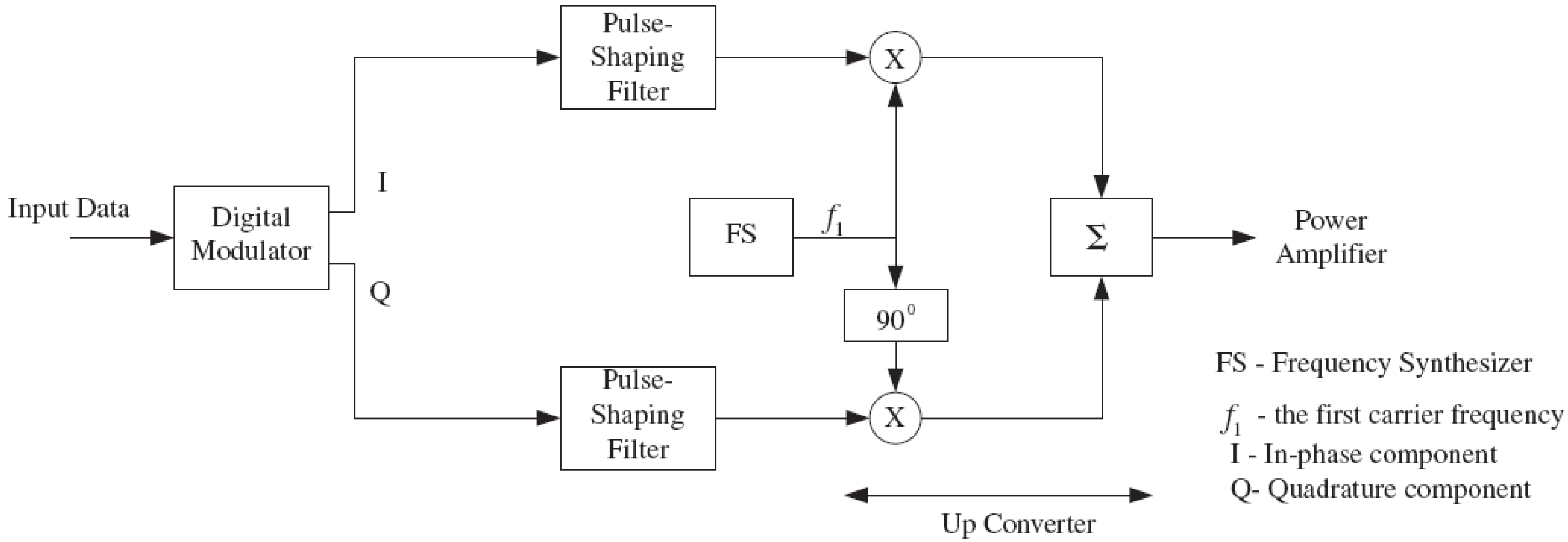
1. Solve Multipath → Multicarrier modulation (FDM)
2. Gain Spectral Efficiency → Orthogonality of the carriers
3. Achieve Efficient Implementation → FFT and IFFT

Chapter 5
OFDM

5.2 Multi-Carrier Transmission

Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

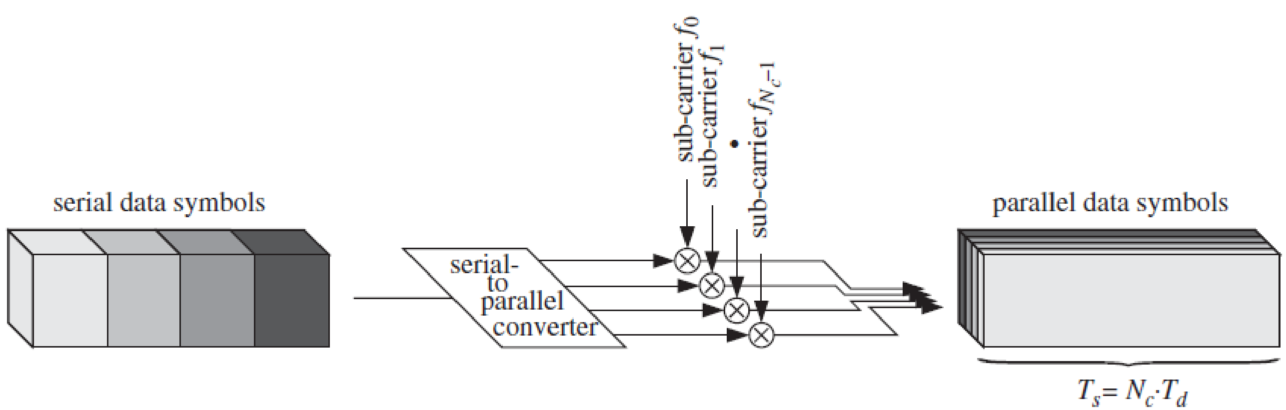
Single-Carrier Transmission



[Karim and Sarraf, 2002, Fig 3-1]

Multi-Carrier Transmission

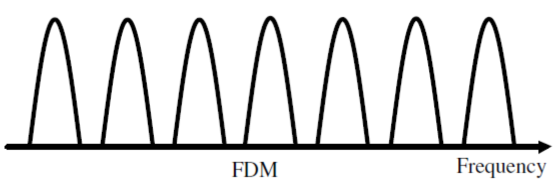
- Convert a serial high rate data stream on to multiple parallel low rate sub-streams.
- Each sub-stream is modulated on its own sub-carrier.
- Since the symbol rate on each sub-carrier is much less than the initial serial data symbol rate, the effects of delay spread, i.e. ISI, significantly decrease, reducing the complexity of the equalizer.



[Fazel and Kaiser, 2008, Fig 1-4]

Frequency Division Multiplexing

- To facilitate separation of the signals at the receiver, the carrier frequencies were spaced sufficiently far apart so that the signal spectra did not overlap. Empty spectral regions between the signals assured that they could be separated with readily realizable filters.
- The resulting spectral efficiency was therefore quite low.



Multi-Carrier (FDM) vs. Single Carrier

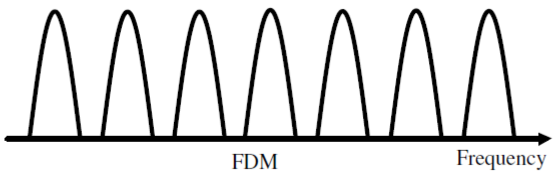
Single Carrier	Multi-Carrier (FDM)
Single higher rate serial scheme	Parallel scheme. Each of the parallel subchannels can carry a low signalling rate, proportional to its bandwidth.
❍ Multipath problem: Far more susceptible to inter-symbol interference (ISI) due to the short duration of its signal elements and the higher distortion produced by its wider frequency band ❍ Complicated equalization	❍ Long duration signal elements and narrow bandwidth in sub-channels. ❍ Complexity problem: If built straightforwardly as several (*N*) transmitters and receivers, will be more costly to implement. ❍ BW efficiency problem: The sum of parallel signalling rates is less than can be carried by a single serial channel of that combined bandwidth because of the unused guard space between the parallel sub-carriers.

FDM (con't)

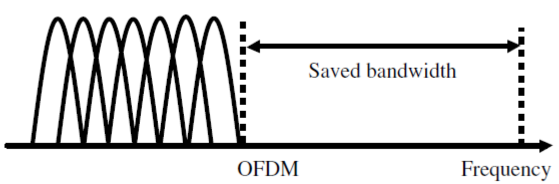
- Before the development of equalization, the parallel technique was the preferred means of achieving high rates over a dispersive channel, in spite of its high cost and relative bandwidth inefficiency.

OFDM

- OFDM = Orthogonal frequency division multiplexing
- One of multi-carrier modulation (MCM) techniques
  - Parallel data transmission (of many sequential streams)
  - A broadband is divided into many narrow sub-channels
  - Frequency division multiplexing (FDM)



- High spectral efficiency
  - The sub-channels are made orthogonal to each other over the OFDM symbol duration T_s.
    - Spacing is carefully selected.
  - Allow the sub-channels to overlap in the frequency domain.
  - Allow sub-carriers to be spaced as close as theoretically possible.



Orthogonality

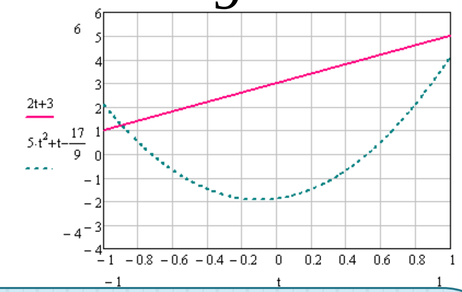
- Two vectors/functions are **orthogonal** if their **inner product** is zero.
- The symbol $\perp$ is used to denote orthogonality.

Vector:

$$\langle \vec{a},\vec{b}\rangle = \vec{a}\cdot\vec{b}^{*} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\cdot\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} a_k b_k = 0$$

Example:

$$2t+3 \text{ and } 5t^2+t-\frac{17}{9} \text{ on } [-1,1]$$



Time-domain:

$$\langle a,b\rangle = \int_{-\infty}^{\infty} a(t)\,b^{*}(t)\,dt = 0$$

Complex conjugate

Frequency domain:

$$\langle A,B\rangle = \int_{-\infty}^{\infty} A(f)\,B^{*}(f)\,df = 0$$

Example:

$$\sin\left(2\pi k_1 \frac{t}{T}\right) \text{ and } \cos\left(2\pi k_2 \frac{t}{T}\right) \text{ on } [0,T]$$

$$e^{j2\pi n\frac{t}{T}} \text{ on } [0,T]$$

Orthogonality in Communication

CDMA

$$s(t)=\sum_{k=0}^{\ell-1} S_k c_k(t) \xrightarrow{\mathcal{F}} S(f)=\sum_{k=0}^{\ell-1} S_k C_k(f) \qquad \text{where} \quad c_{k_1} \perp c_{k_2}$$

TDMA

$$s(t)=\sum_{k=0}^{\ell-1} S_k c(t-kT_s) \xrightarrow{\mathcal{F}} S(f)=C(f)\sum_{k=0}^{\ell-1} S_k e^{-j2\pi fkT_s}$$

where $c(t)$ is time-limited to $[0,T]$.

This is a special case of CDMA with $c_k(t)=c(t-kT_s)$

The c_k are non-overlapping in time domain.

FDMA

$$S(f)=\sum_{k=0}^{\ell-1} S_k C(f-k\Delta f)$$

where $C(f)$ is frequency-limited to $[0,\Delta f]$.

This is a special case of CDMA with $C_k(f)=C(f-k\Delta f)$

The C_k are non-overlapping in freq. domain.

OFDM

- Let $S_1, S_2, \ldots, S_N$ be the information symbol.
- The discrete baseband OFDM modulated symbol can be expressed as

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kt}{T_s} \right), \quad 0 \le t \le T_s$$

$$= \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left(j\frac{2\pi kt}{T_s} \right)}_{c_k(t)}$$

Another special case of CDMA!

Note that:

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\mathrm{Re}\{S_k\} \cos\left(\frac{2\pi kt}{T_s} \right) - \mathrm{Im}\{S_k\} \sin\left(\frac{2\pi kt}{T_s} \right) \right)$$

OFDM: Orthogonality

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \int_0^{T_s} \exp\left(j\frac{2\pi k_1 t}{T_s}\right)\exp\left(-j\frac{2\pi k_2 t}{T_s}\right)dt$$

$$= \int_0^{T_s} \exp\left(j\frac{2\pi(k_1 - k_2)t}{T_s}\right)dt = \begin{cases} T_s, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases}$$

When $k_1 = k_2$,

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \int_0^{T_s} 1dt = T_s$$

When $k_1 \neq k_2$,

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \frac{T_s}{j2\pi(k_1 - k_2)}\exp\left(j\frac{2\pi(k_1 - k_2)t}{T_s}\right)\Bigg|_0^{T_s}$$

$$= \frac{T_s}{j2\pi(k_1 - k_2)}(1-1) = 0$$

Frequency Spectrum

$$s(t) = \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left(j\frac{2\pi kt}{T_s} \right)}_{c_k(t)}$$

$$\Delta f = \frac{1}{T_s}$$

This is the term that makes the technique FDM.

$$1_{\left[-\frac{T_s}{2},\frac{T_s}{2}\right]}(t) \xrightarrow{\mathcal{F}} T_s \operatorname{sinc}(\pi T_s f)$$

$$c(t) = \frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \xrightarrow{\mathcal{F}} C(f) = \frac{1}{\sqrt{N}} T_s e^{-j2\pi f\frac{T_s}{2}} \operatorname{sinc}(\pi T_s f)$$

$$c_k(t) = c(t)\exp\left(j\frac{2\pi kt}{T_s} \right) \xrightarrow{\mathcal{F}} C_k(f) = C\left(f - \frac{k}{T_s} \right) = C(f - k\Delta f)$$

$$s(t) = \sum_{k=0}^{N-1} S_k c_k(t) \xrightarrow{\mathcal{F}} S(f) = \sum_{k=0}^{N-1} S_k C_k(f)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k e^{-j2\pi(f-k\Delta f)\frac{T_s}{2}} T_s \operatorname{sinc}\left(\pi T_s (f - k\Delta f)\right)$$

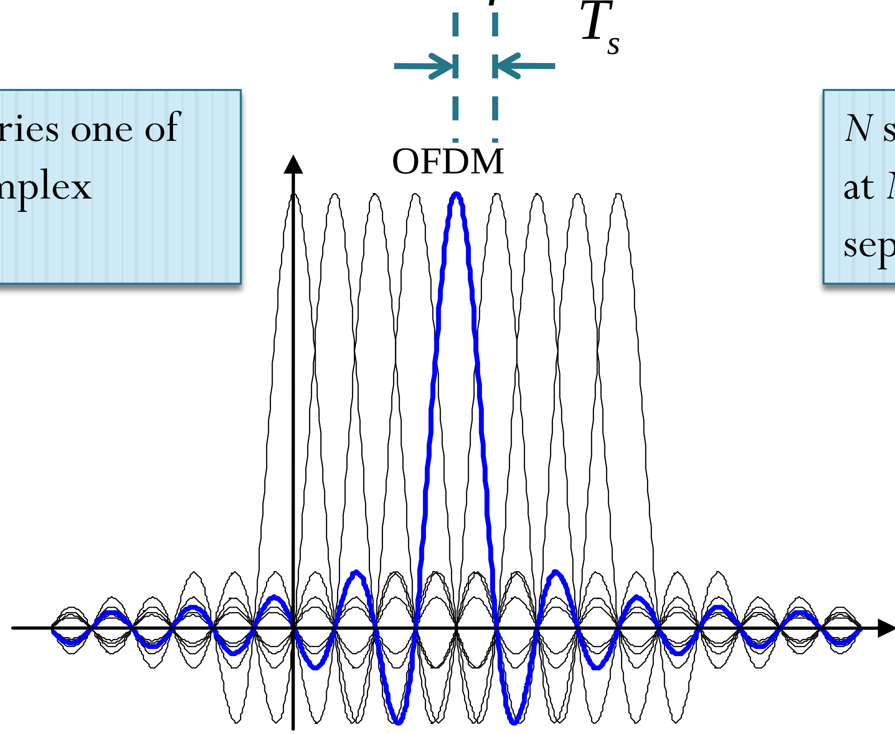
Subcarrier Spacing

$$s(t)=\sum_{k=0}^{N-1}S_k\frac{1}{\sqrt{N}}1_{[0,T_s]}(t)\exp\left(j\frac{2\pi kt}{T_s}\right)$$

$$S(f)=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}S_k e^{-j2\pi(f-k\Delta f)\frac{T_s}{2}}T_s\,\mathrm{sinc}\left(\pi T_s\left(f-k\Delta f\right)\right)$$

$$\Delta f=\frac{1}{T_s}$$

Each QAM signal carries one of the original input complex numbers.



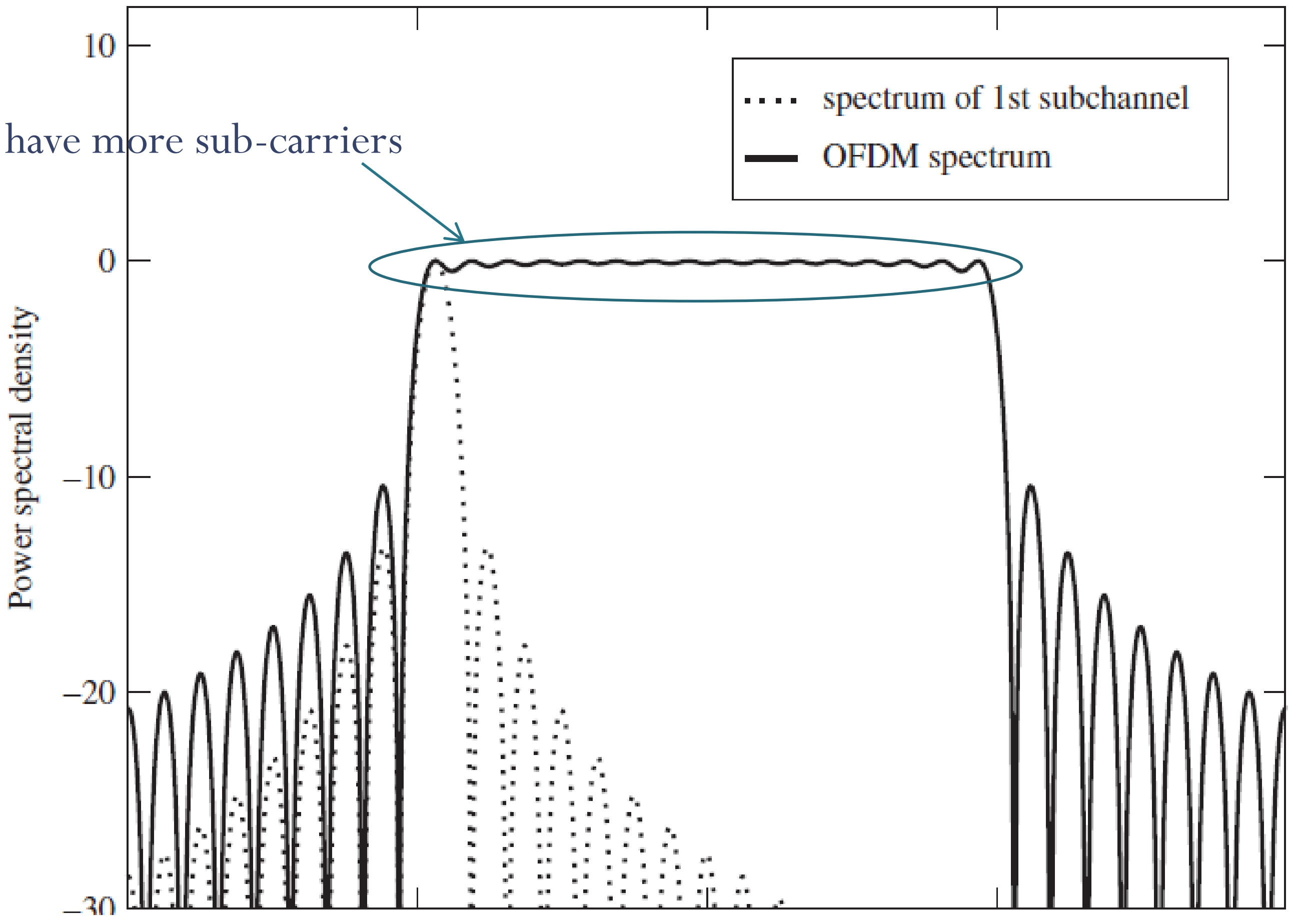
Spectrum Overlap in OFDM

N separate QAM signals, at N frequencies separated by the signalling rate.

The spectrum of each QAM signal is of the form with nulls at the center of the other sub-carriers.

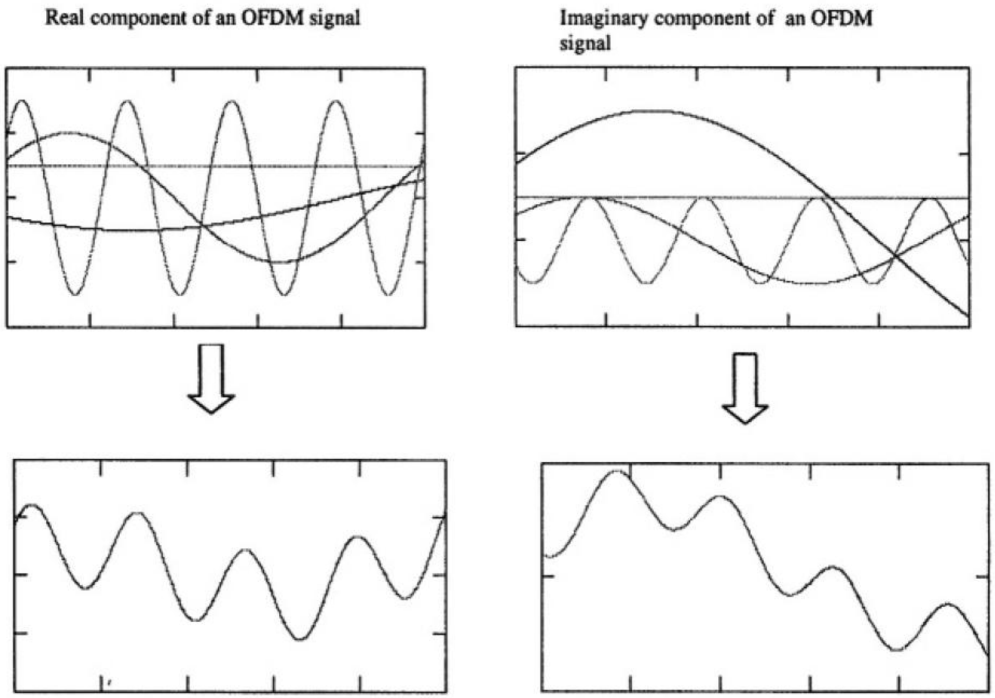
Normalized Power Density Spectrum

Flatter when have more sub-carriers



[Fazel and Kaiser, 2008, Fig 1-5]

Time-Domain Signal



[Bahai, 2002, Fig 1.7]

Real and Imaginary components of an OFDM symbol is the *superposition of several harmonics* modulated by data symbols

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kt}{T_s} \right), \quad 0 \le t \le T_s$$

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\underbrace{\mathrm{Re}\{S_k\} \cos\left(\frac{2\pi kt}{T_s} \right)}_{\text{in-phase part}} - \underbrace{\mathrm{Im}\{S_k\} \sin\left(\frac{2\pi kt}{T_s} \right)}_{\text{quadrature part}} \right)$$

Summary

- So, we have a scheme which achieve
  - Large symbol duration (T_s) and hence less multipath problem
  - Good spectral efficiency
- One more problem:
  - There are so many carriers!

Chapter 5
OFDM

5.3 DFT and FFT

Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

Discrete Fourier Transform (DFT)

Transmitter produces

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kt}{T_s} \right), \quad 0 \le t \le T_s$$

Sample the signal in time domain every T_s / N gives

$$s[n] = s\left(n \frac{T_s}{N} \right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{N} \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kn}{N} \right) = \sqrt{N}\, \mathrm{IDFT}\{S\}[n]$$

We can implement OFDM in the discrete domain!

Discrete Fourier Transform (DFT)

In DFT, we work with N-point signal (finite-length sequence of length N) in both time and frequency domain. To simplify the definition we define

$$\psi_N = e^{j\frac{2\pi}{N}}$$

and the DFT matrix $Q = \Psi_N$ whose element on the pth row and qth column is given by $\psi_N^{-(p-1)(q-1)}$:

The "-1" are there because we start from row 1 and column 1.

$$\Psi_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \psi_N^{-1} & \psi_N^{-2} & \cdots & \psi_N^{-(N-1)} \\ 1 & \psi_N^{-2} & \psi_N^{-4} & \cdots & \psi_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \cdots & \psi_N^{-(N-1)(N-1)} \end{bmatrix}$$

Key Property:

$\Psi_N^{-1} = \frac{1}{N}\Psi_N^*$. Equivalently, $\Psi_N^{-1}\Psi_N = NI_N$.

$\frac{1}{\sqrt{N}}\Psi_N$ is a unitary matrix

DFT

Definition 5.3. The N-point DFT of an N-point signal (column vector) x is given by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jnk\frac{2\pi}{N}} = \boxed{\sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}} \;; 0 \leq k < N.$$

The inverse DFT is given by

$$\boxed{\underset{0\leq n<N}{x[n]} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, \psi_N^{nk}} \underset{\text{DFT}^{-1}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\leq k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}$$

In matrix form,

$$\boxed{x = \frac{1}{N}\Psi_N^* X \underset{\text{DFT}^{-1}}{\overset{\text{DFT}}{\rightleftharpoons}} X = \Psi_N \times x.}$$

DFT

Definition 5.3. The N-point DFT of an N-point signal (column vector) x is given by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jnk\frac{2\pi}{N}} = \boxed{\sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}} \ ; 0 \le k < N.$$

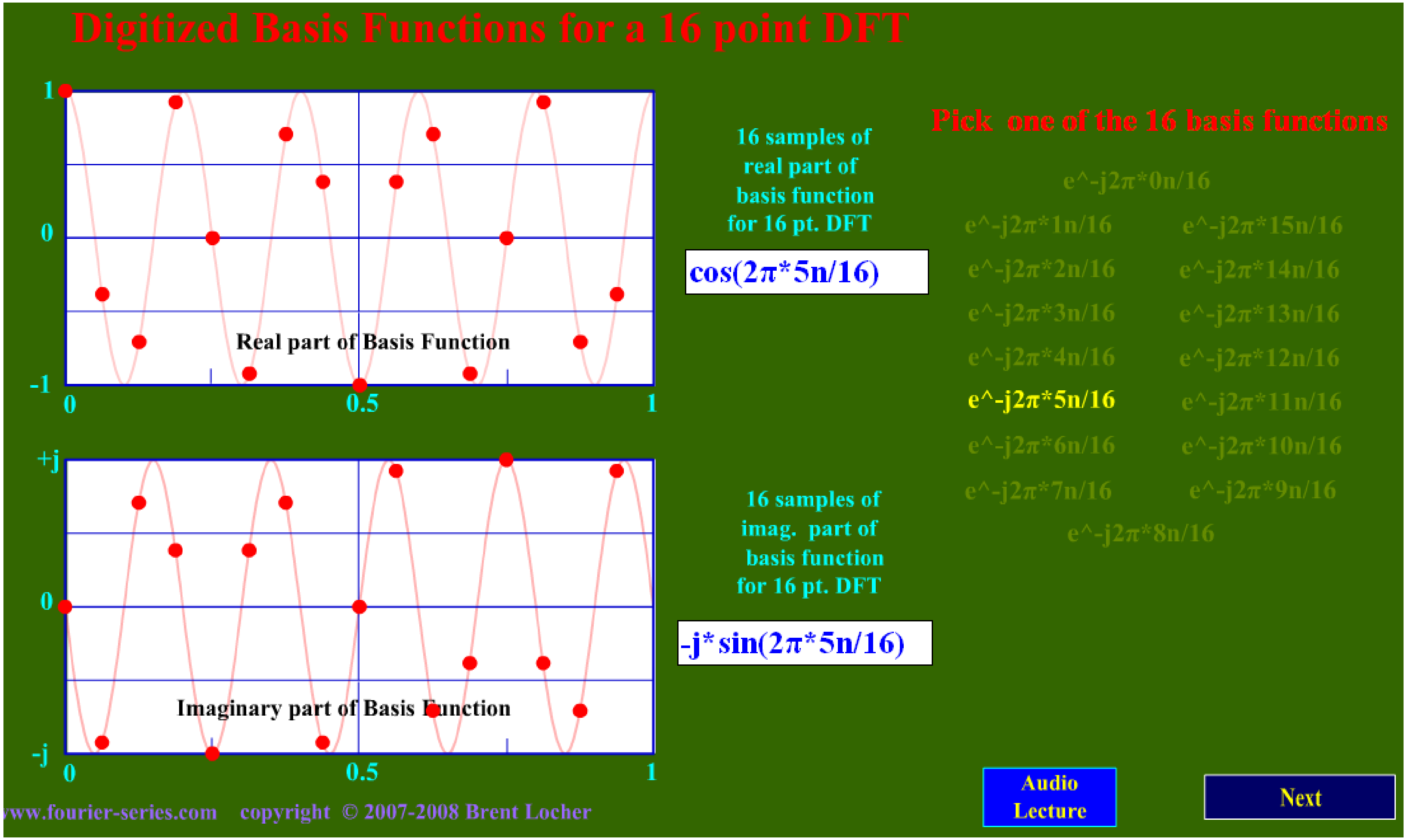
The inverse DFT is given by

$$\boxed{\underset{0\le n<N}{x[n]} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, \psi_N^{nk}} \underset{\text{DFT}^{-1}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}$$

In matrix form,

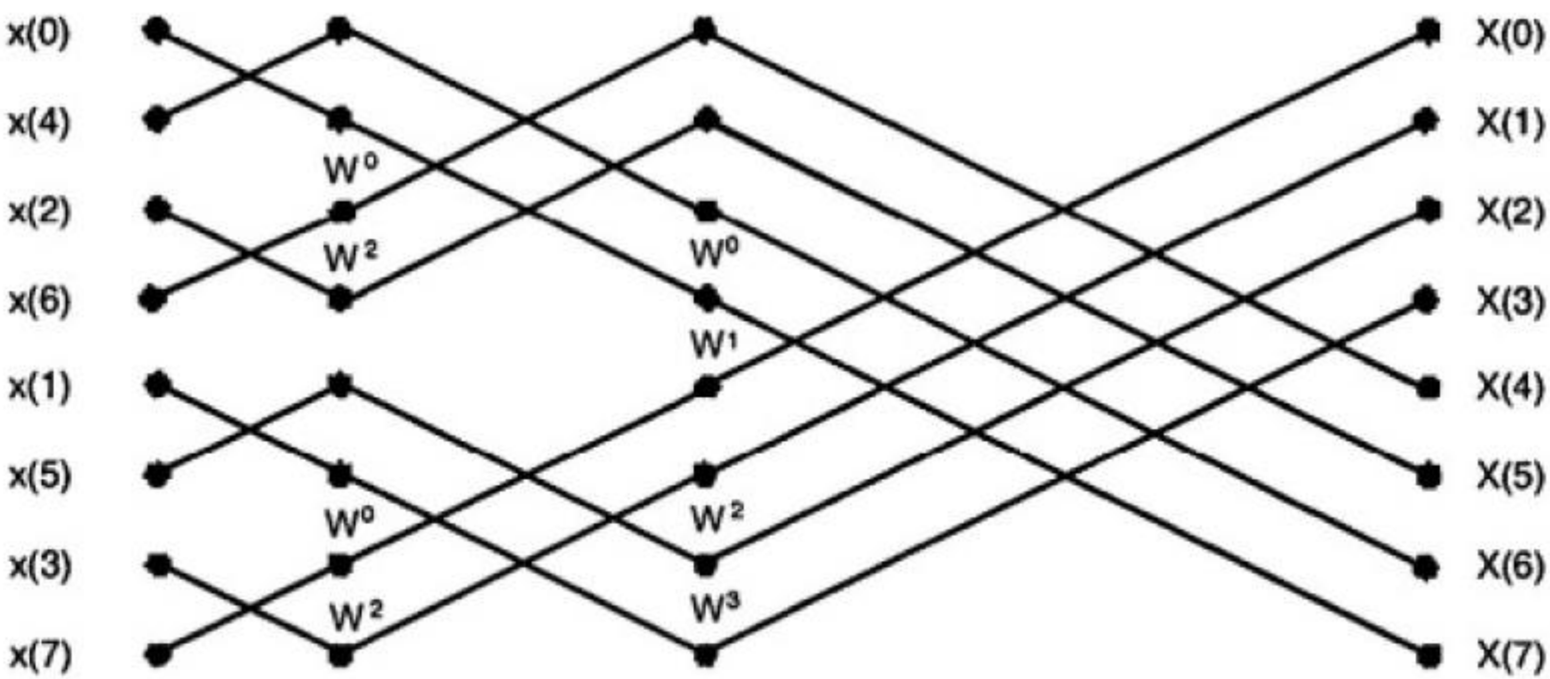
$$\boxed{x = \frac{1}{N}\Psi_N^{*} X \underset{\text{DFT}^{-1}}{\overset{\text{DFT}}{\rightleftharpoons}} X = \Psi_N \times x.}$$

DFT: Example



[http://www.fourier-series.com/fourierseries2/DFT_tutorial.html]

Efficient Implementation: (I)FFT



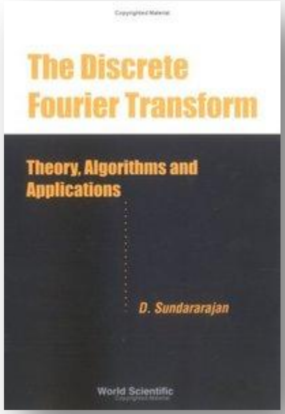
[Bahai, 2002, Fig. 2.9]

An N-point FFT requires only on the order of $N \log N$ multiplications, rather than N^2 as in a straightforward computation.

FFT

- The history of the FFT is complicated.
- As with many discoveries and inventions, it arrived before the (computer) world was ready for it.
- Usually done with N a power of two.
  - Not only is it very efficient in terms of computing time, but is ideally suited to the binary arithmetic of digital computers.
  - From the implementation point of view it is better to have, for example, a FFT size of 1024 even if only 600 outputs are used than try to have another length for FFT between 600 and 1024.

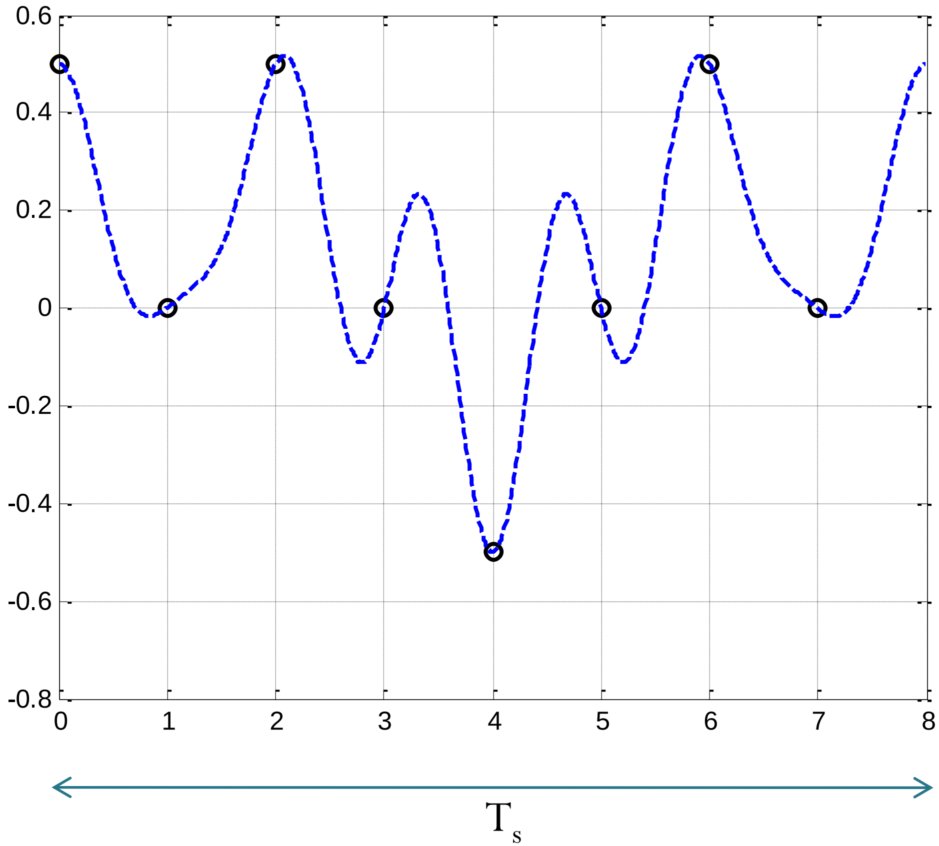
References: E. Oran Brigham, *The Fast Fourier Transform*, Prentice-Hall, 1974.



DFT Samples

$$s(t)=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}S_k \exp\left(j\frac{2\pi kt}{T_s}\right), \quad 0\le t\le T_s$$

- Here are the points s[n] on the continuous-time version s(t):



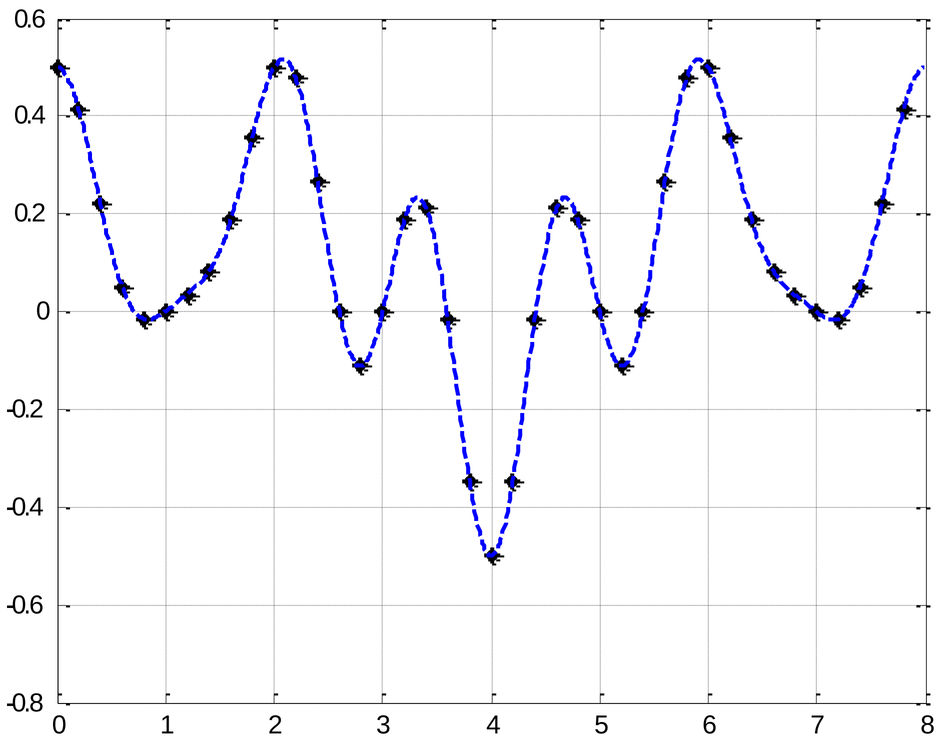
$$s[n]=s\left(n\frac{T_s}{N}\right)$$

$$=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}S_k \exp\left(j\frac{2\pi kn}{N}\right)$$

$$=\sqrt{N}\,\mathrm{IDFT}\{S\}[n]$$

$$0\le n< N$$

Oversampling



Oversampling (2)

- Increase the number of sample points from N to LN on the interval $[0, T_s]$.
- ***L*** is called the **over-sampling factor**.

$$s[n] = s\left(n\frac{T_s}{N}\right)$$
$$0 \le n < N$$

→

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right)$$
$$0 \le n < LN$$

$$s^{(L)}[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{LN}\right) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)$$

$$= \frac{1}{\sqrt{N}} LN \left(\frac{1}{LN}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\left(\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right) + \sum_{k=N}^{NL-1} 0 \exp\left(j\frac{2\pi kn}{LN}\right)\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\sum_{k=0}^{NL-1} \tilde{S}_k \exp\left(j\frac{2\pi kn}{LN}\right)\right) = L\sqrt{N}\ \text{IDFT}\left\{\tilde{S}\right\}[n]$$

Zero padding:

$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$

Oversampling: Summary

N points

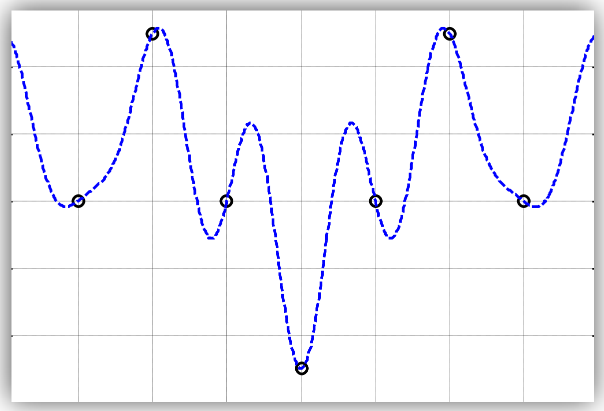
$$s[n] = s\left(n\frac{T_s}{N}\right) = \sqrt{N}\,\mathrm{IDFT}\{S\}[n]$$
$$0 \le n < N$$

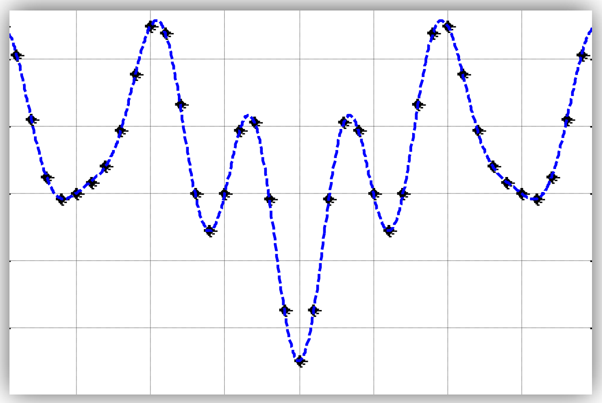
LN points

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) = L\sqrt{N}\,\mathrm{IDFT}\{\tilde{S}\}[n]$$
$$0 \le n < LN$$

Zero padding:

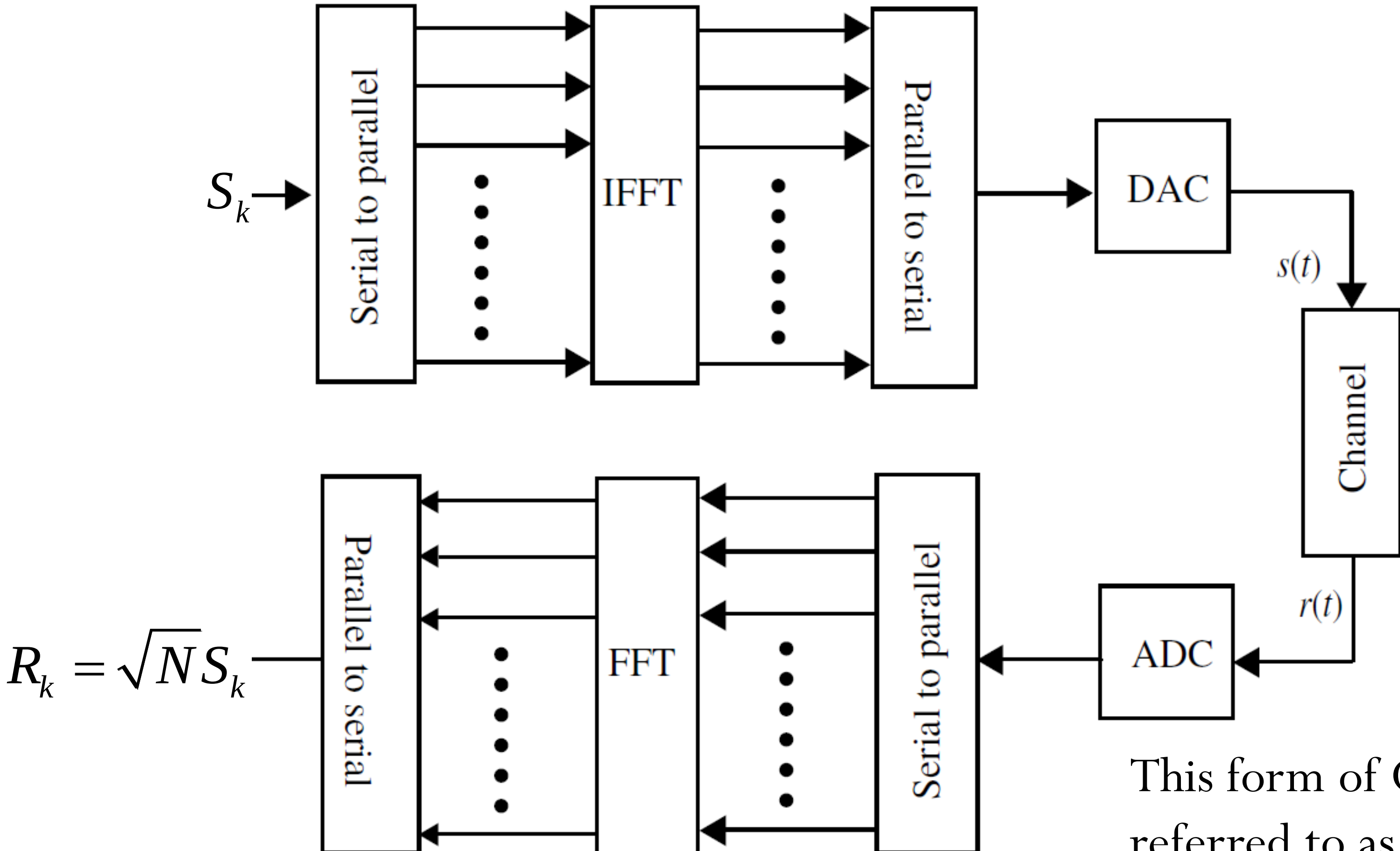
$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$





OFDM implementation by IFFT/FFT

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) = L\sqrt{N}\,\mathrm{IFFT}^{(L)}\{\tilde{S}\}[n]$$



$$r[n] = \sqrt{N}\,\mathrm{IFFT}\{S\}[n]$$

This form of OFDM is often referred to as **Discrete Multi-Tone (DMT)**.

OFDM with Memoryless Channel

$$h(t) = \beta\delta(t)$$

$[\text{should be } h(t) = \beta\delta(t-\tau)]$

$$y(t) = h(t) * s(t) + w(t) = \beta s(t) + w(t)$$

Additive white Gaussian noise

Sample every T_s/N

$$y[n] = \beta s[n] + w[n]$$

FFT

$$s[n] = \sqrt{N}\,\text{IFFT}\{S\}[n]$$

$$Y_k = \text{FFT}\{y\}[n] = \beta\sqrt{N}S_k + W_k$$

Sub-channel are independent.

(No ICI)

Channel with Finite Memory

Discrete time baseband model:

$$y[n] = \{h * s\}[n] + w[n] = \sum_{m=0}^{\nu} h[m] s[n-m] + w[n]$$

[Tse Viswanath, 2005, Sec. 2.2.3]

where $h[n] = 0$ for $n < 0$ and $n > \nu$

$$w[n] \overset{i.i.d.}{\sim} \mathcal{CN}(0, N_0)$$

We will assume that $\nu \ll N$

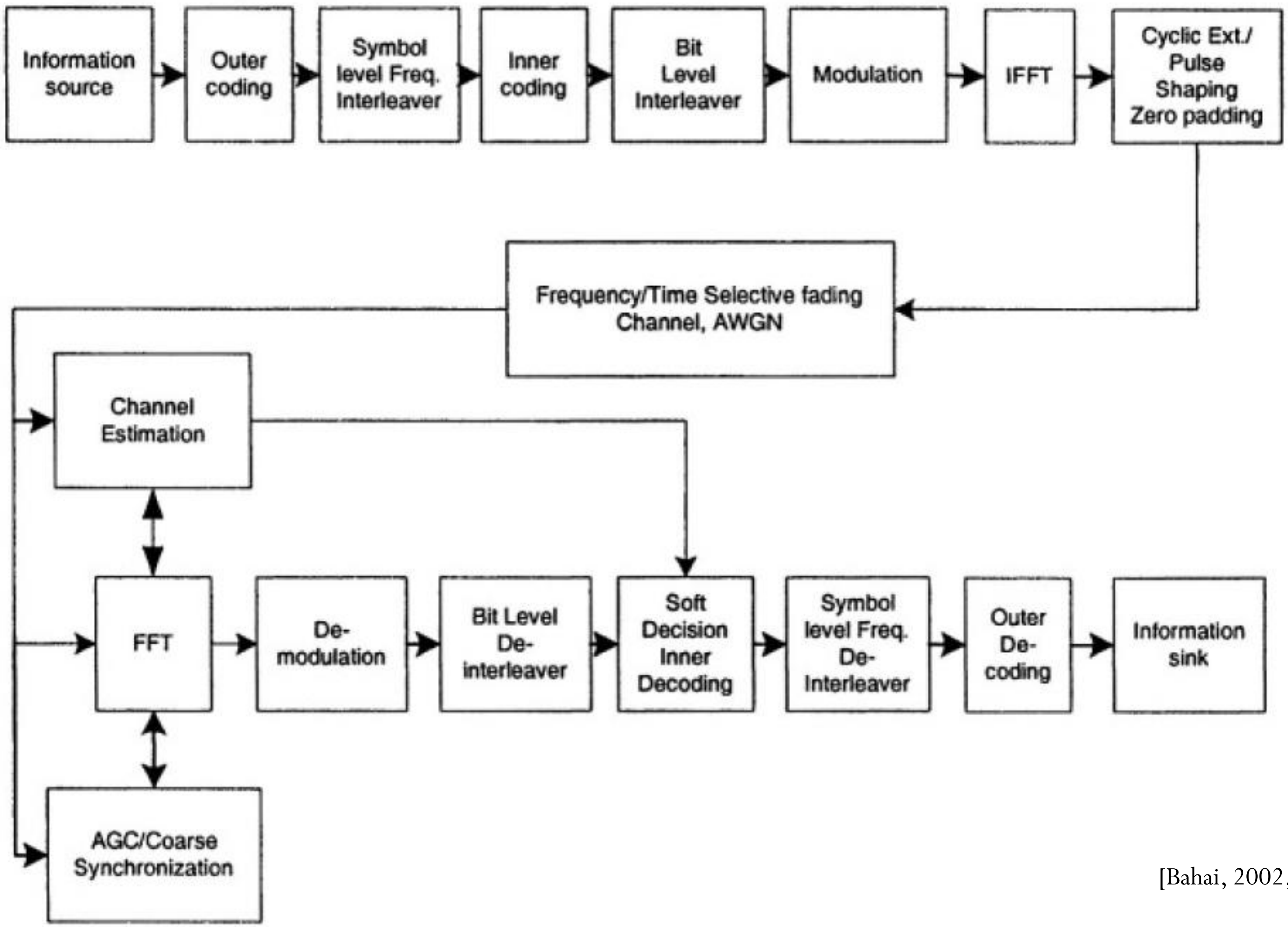
Remarks:

$Z = X + jY$ is a ***complex Gaussian*** if X and Y are jointly Gaussian.

If X, Y is i.i.d. $\mathcal{N}(0, \sigma^2)$, then $Z = X + iY \sim \mathcal{CN}(0, \sigma_Z^2)$ where $\sigma_Z^2 = 2\sigma^2$ with

$$f_Z(z) = f_{X,Y}(\mathrm{Re}\{z\}, \mathrm{Im}\{z\}) = \frac{1}{\pi\sigma_Z^2} e^{-\frac{|z|^2}{\sigma_Z^2}}.$$

OFDM Architecture



[Bahai, 2002, Fig 1.11]

Chapter 5
OFDM

5.4 Cyclic Prefix (CP)

Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

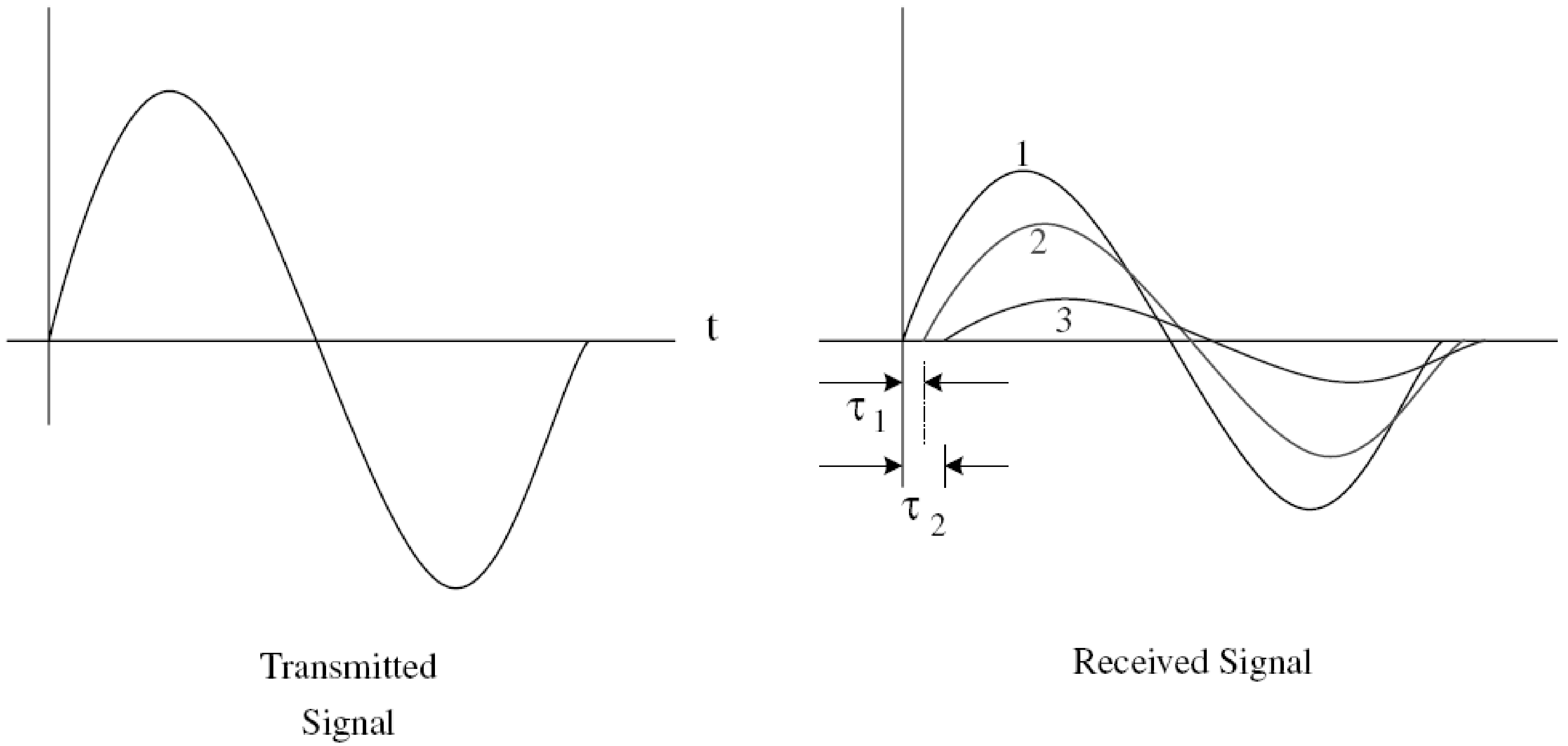
Three steps towards modern OFDM

1. Mitigate Multipath (ISI) → Multicarrier modulation (FDM)
2. Gain Spectral Efficiency → Orthogonality of the carriers
3. Achieve Efficient Implementation → FFT and IFFT

- Completely eliminate ISI and ICI → Cyclic prefix

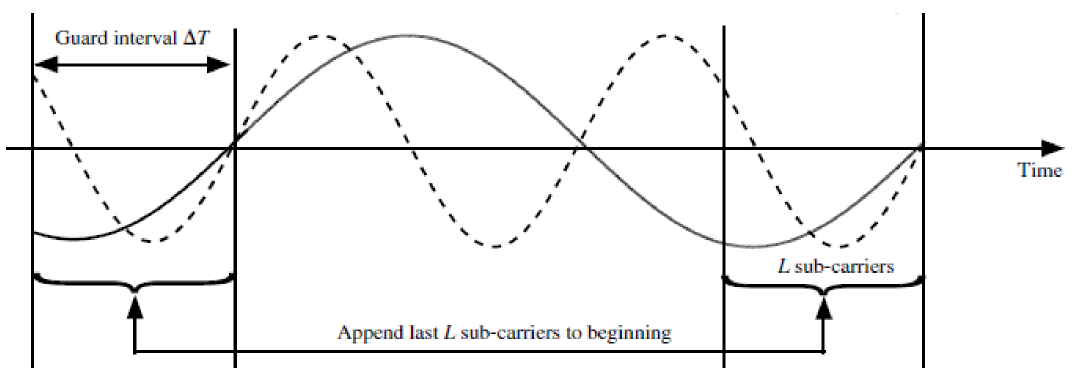
Cyclic Prefix: Motivation (1)

- Recall: Multipath Fading and Delay Spread



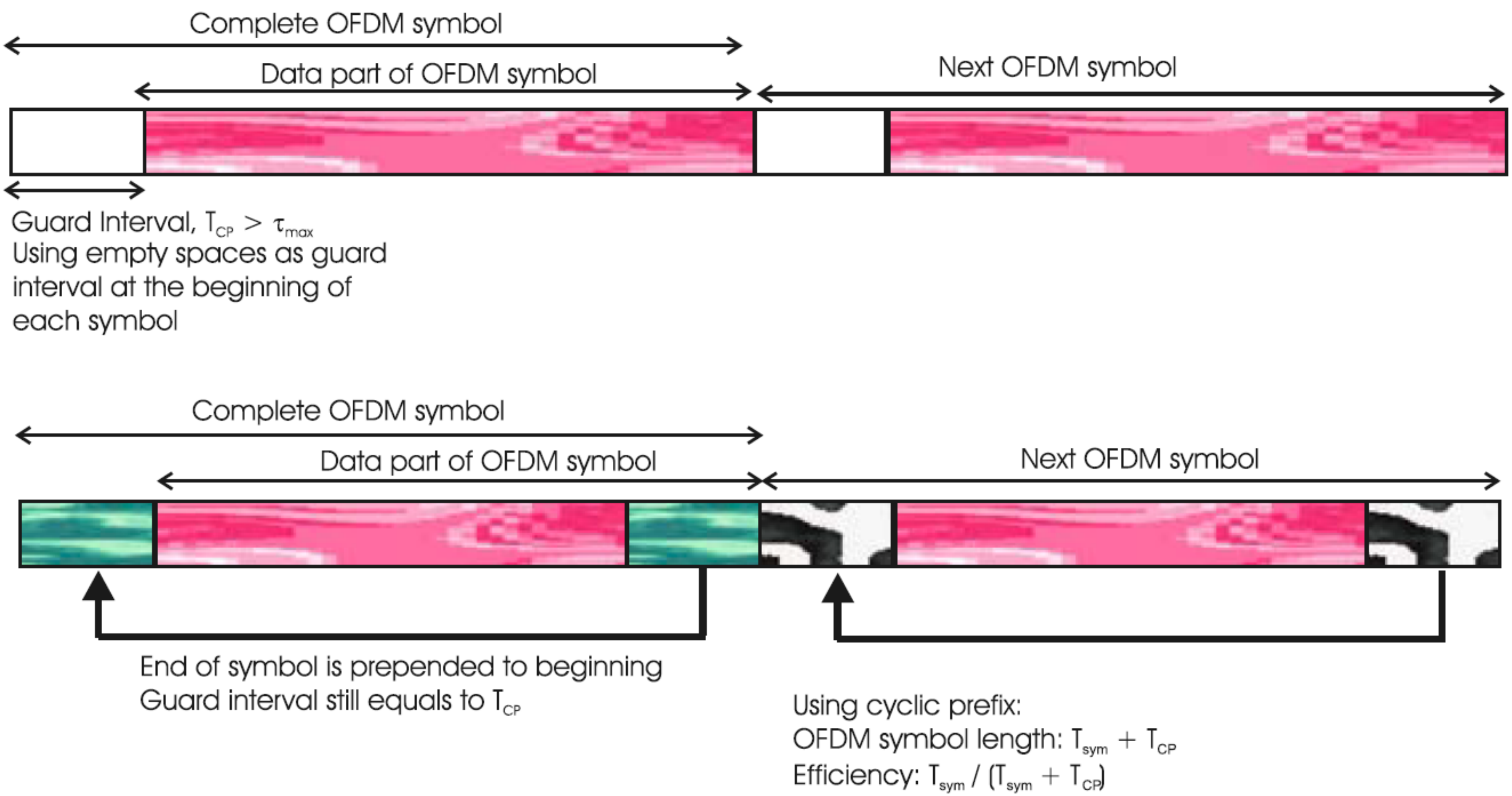
Cyclic Prefix: Motivation (2)

- When the number of sub-carriers increases, the OFDM symbol duration T_s becomes large compared to the duration of the impulse response τ_{max} of the channel, and the amount of ISI reduces.
- Can we "eliminate" the multipath (**ISI**) problem?
- To reduce the ISI, add **guard interval** larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., **ICI** (inter-channel interference) still exists.
- To prevent both the ISI as well as the ICI, OFDM symbol is **cyclically extended** into the guard interval.



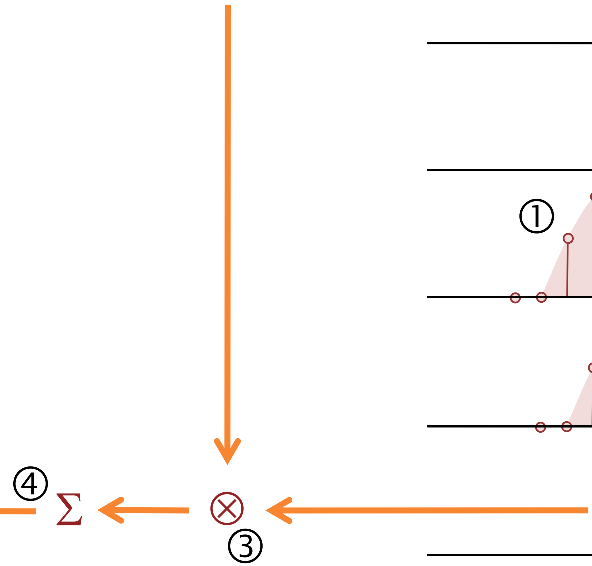
[Jiang]

Cyclic Prefix

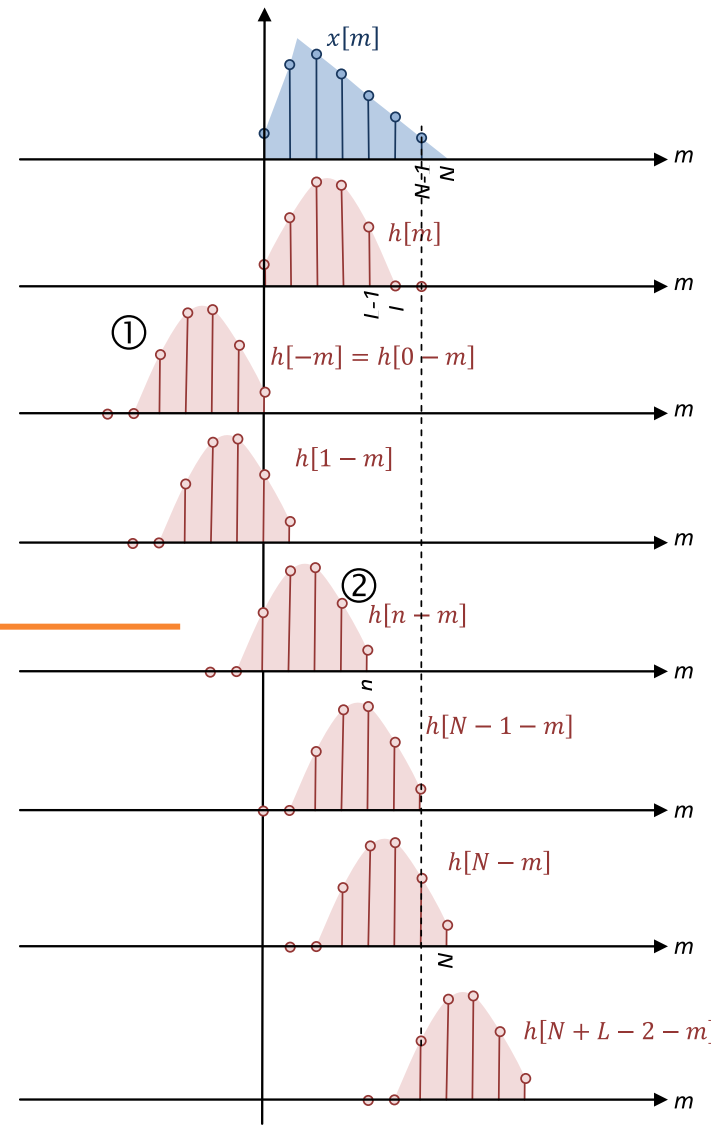


Convolution

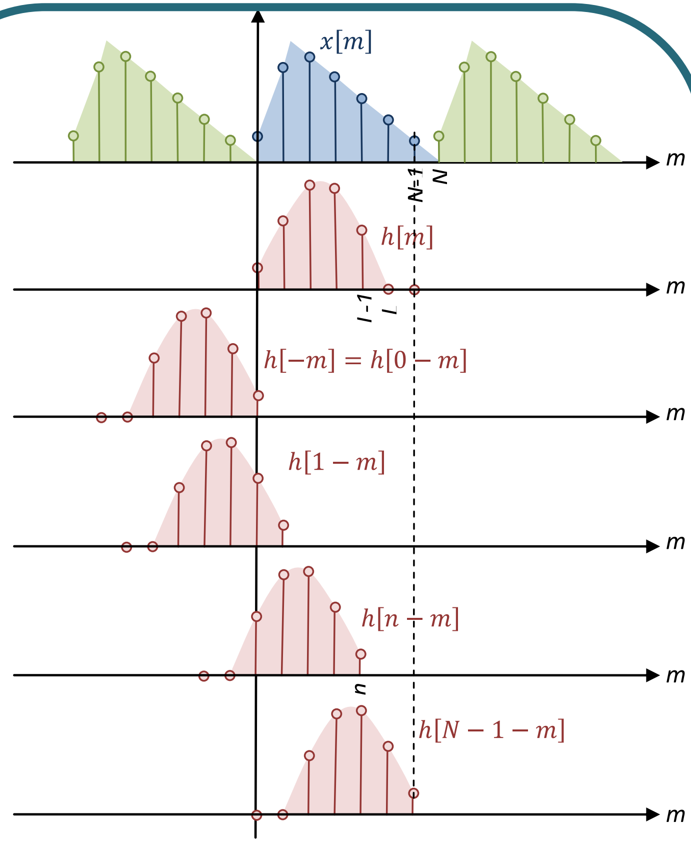
① Flip
② Shift
③ Multiply
④ Add



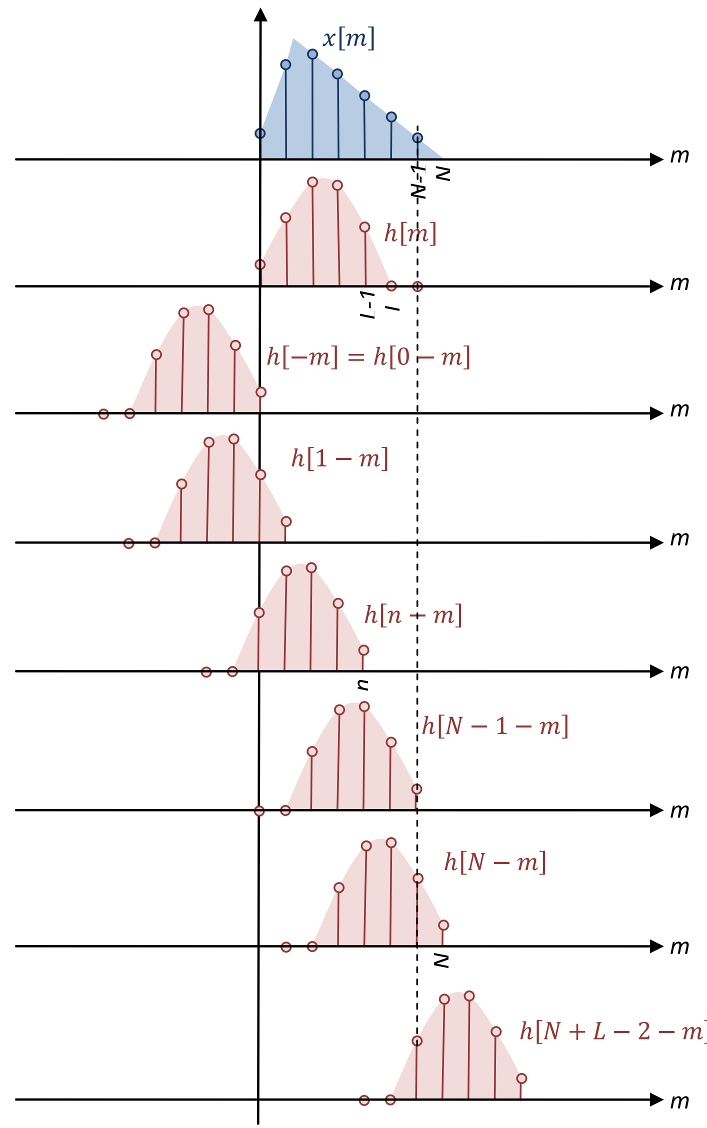
$$\{x * h\}[n] = \sum_{m} x[m]h[n-m]$$



Circular Convolution



Replicate x (now it looks periodic)
Then, perform the usual convolution only on n = 0 to N-1



Circular Convolution: Example

Find

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix} * \begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix} \circledast \begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3 & 0 & 0\end{bmatrix} \circledast \begin{bmatrix}4 & 5 & 6 & 0 & 0\end{bmatrix}$$

Discussion

- Circular convolution can be used to find the regular convolution by zero-padding.
- In modern OFDM, it is another way around.
- CTFT: convolution in time domain corresponds to multiplication in frequency domain.
- DFT: circular convolution in (discrete) time domain corresponds to multiplication in (discrete) frequency domain.
  - We want to have multiplication in frequency domain.
  - So, we want circular convolution and not the regular convolution.
- Real channel does regular convolution.
- With cyclic prefix, regular convolution can be used to create circular convolution.

Example

- Suppose $x^{(1)}$ = [1 -2 3 1 2] and h = [3 2 1]
- [1 -2 3 1 2] ⊛ [3 2 1 0 0] = [8 -2 6 7 11]
- [1 2 1 -2 3 1 2] * [3 2 1] = [3 8 8 -2 6 7 11 5 2]
- Suppose $x^{(2)}$ = [2 1 -3 -2 1]
- [2 1 -3 -2 1] ⊛ [3 2 1 0 0] = [6 8 -5 -11 -4]
- [-2 1 2 1 -3 -2 1] * [3 2 1] = [-6 -1 6 8 -5 -11 -4 0 1]
- [1 2 1 -2 3 1 2 -2 1 2 1 -3 -2 1] * [3 2 1] =

[3 8 8 -2 6 7 11 5 2]

+

[-6 -1 6 8 -5 -11 -4 0 1]

= [3 8 8 -2 6 7 11 -1 1 6 8 -5 -11 -4 0 1]

Circular Convolution: Key Properties

- Consider an N-point signal $x[n]$
- **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last ν samples of the symbols at the beginning of the symbol:

$$\hat{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ x[n+N], & -v \le n \le -1 \end{cases}$$

- Key Property 1:

$$\{h \circledast x\}[n] = (h * \hat{x})[n] \text{ for } 0 \le n \le N-1$$

- Key Property 2:

$$\{h \circledast x\}[n] \xrightarrow{\text{FFT}} H_k X_k$$

OFDM with CP for Channel w/ Memory

- We want to send N samples $S_0, S_1, \ldots, S_{N-1}$ across noisy channel with memory.
- First apply IFFT: $S_k \xrightarrow{\text{IFFT}} s[n]$
- Then, add cyclic prefix

$$\hat{s} = \left[s[N-v], \ldots, s[N-1], s[0], \ldots, s[N-1] \right]$$

- This is inputted to the channel.
- The output is

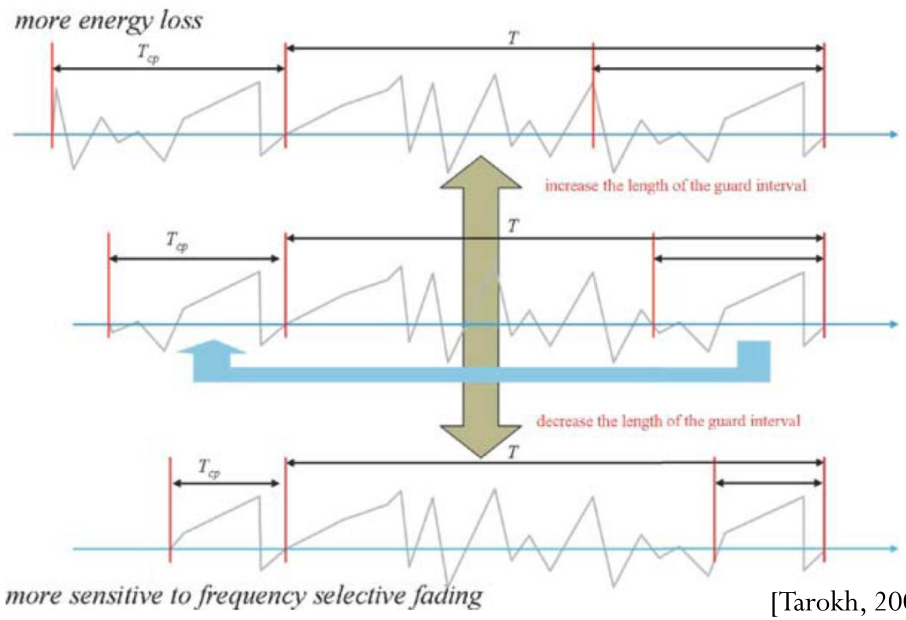
$$y[n] = \left[p[N-v], \ldots, p[N-1], r[0], \ldots, r[N-1] \right]$$

- Remove cyclic prefix to get $r[n] = h[n] \circledast s[n] + w[n]$
- Then apply FFT: $r[n] \xrightarrow{\text{FFT}} R_k$
- By circular convolution property of DFT, $R_k = H_k S_k + W_k$

No ICI!

OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.



[Tarokh, 2009, Fig 2.9]

Chapter 5
OFDM

5.5 Remarks about OFDM

Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

Summary: OFDM Advantages

- For a given channel delay spread, the implementation complexity is much lower than that of a conventional single carrier system with time domain equalizer.
- Spectral efficiency is high since it uses overlapping orthogonal subcarriers in the frequency domain.
- Modulation and demodulation are implemented using inverse discrete Fourier transform (IDFT) and discrete Fourier transform (DFT), respectively, and fast Fourier transform (FFT) algorithms can be applied to make the overall system efficient.
- Capacity can be significantly increased by adapting the data rate per subcarrier according to the signal-to-noise ratio (SNR) of the individual subcarrier.

Example: 802.11a

Parameter	IEEE 802.11a
Bandwidth	20 MHz
Number of sub-carriers N_c	52 (48 data + 4 pilots) (64 FFT)
Symbol duration T_s	4 µs
Carrier spacing F_s	312.5 kHz $= \frac{1}{4-0.8[\mu s]}$
Guard time T_g	0.8 µs
Modulation	BPSK, QPSK, 16-QAM, and 64-QAM
FEC coding	Convolutional with code rate 1/2 up to 3/4
Max. data rate	54 Mbit/s

Summary: OFDM Drawbacks

- High peak-to-average power ratio (PAPR): The transmitted signal is a superposition of all the subcarriers with different carrier frequencies and high amplitude peaks occur because of the superposition.
- High sensitivity to frequency offset: When there are frequency offsets in the subcarriers, the orthogonality among the subcarriers breaks and it causes intercarrier interference (ICI).
- A need for an adaptive or coded scheme to overcome spectral nulls in the channel: In the presence of a null in the channel, there is no way to recover the data of the subcarriers that are affected by the null unless we use rate adaptation or a coding scheme.

Complex-valued Matrix (1)

1) We shall use A^H to denote the **conjugate transpose** (Hermitian adjoint) of A. (In MATLAB, this is `A'`). If A is a real matrix, this is the same as A^T, the transpose of A.
   a) A is said to be **hermitian** if $A^H = A$.
2) A is said to be **normal** if $A^H A = AA^H$.
   a) All unitary, hermitian and positive definite matrices are normal.
   b) The matrix $\begin{pmatrix} -i & -i & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is normal but not unitary, nor hermitian, nor positive definite.

Complex-valued Matrix (2)

c) ***Spectral decomposition***: Normal matrices are precisely the ones to which the spectral theorem applies: For any normal matrix A, there exists a unitary matrix U such that $A = UDU^H$ where D is the diagonal matrix where the entries are the eigenvalues of A.

   i) Furthermore, any matrix which diagonalizes in this way must be normal.

   ii) The column vectors of U are the eigenvectors of A and they are orthogonal.

   iii) The spectral decomposition is a special case of the Schur decomposition. It is also a special case of the singular value decomposition.

Complex-valued Matrix (3)

3) U is said to be unitary matrix if $U^{-1} = U^H$. This is equivalent to

$\equiv$ U^H is unitary

$\equiv$ $U^H U = U U^H = I$

$\equiv$ U is an orthonormal matrix

In which case,

a) $\langle Ux, Uy \rangle = y^H x = \langle x, y \rangle$. $\|Ux\|^2 = \|x\|^2$.

b) Eigenvalues λ are complex numbers with $|\lambda| = 1$.

c) $|\det(U)| = 1$.

d) U is normal; that is $U^H U = U U^H$. So, we can do spectral decomposition.

Chapter 5

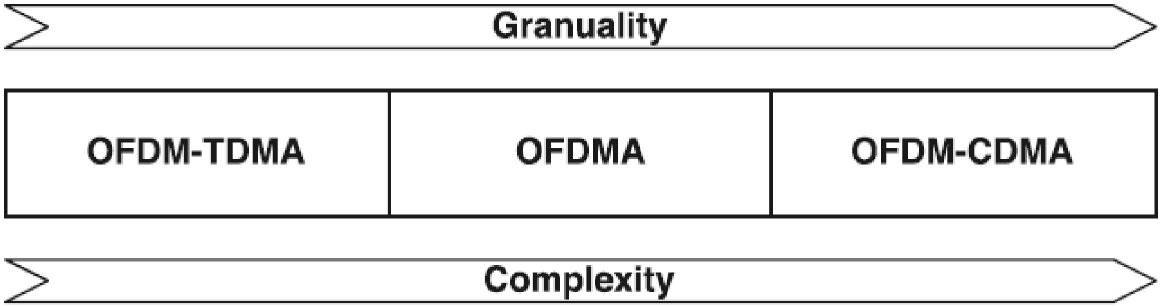
OFDM

5.6 OFDM-Based Multiple Access

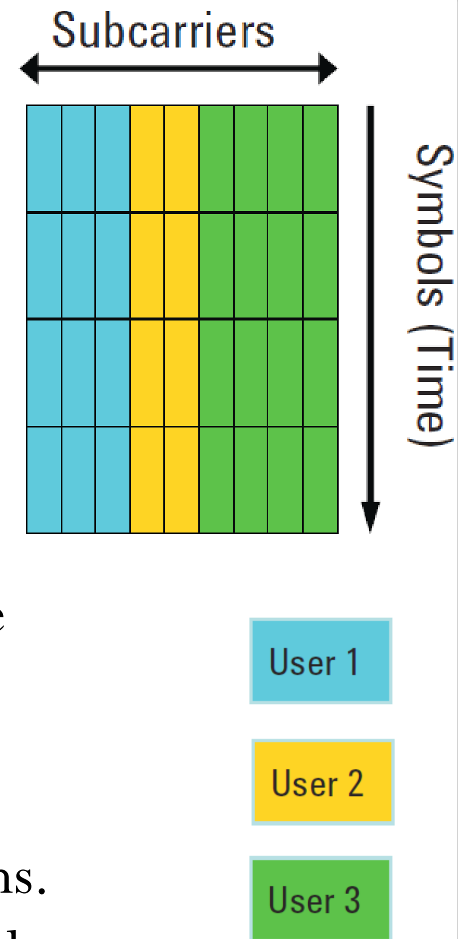
Office Hours:
BKD 3601-7
Tuesday 14:00-16:00
Thursday 9:30-11:30

OFDM-based Multiple Access

- Three multiple access techniques
  1. OFDMA,
  2. OFDM-TDMA, and
  3. OFDM-CDMA



OFDM-TDMA

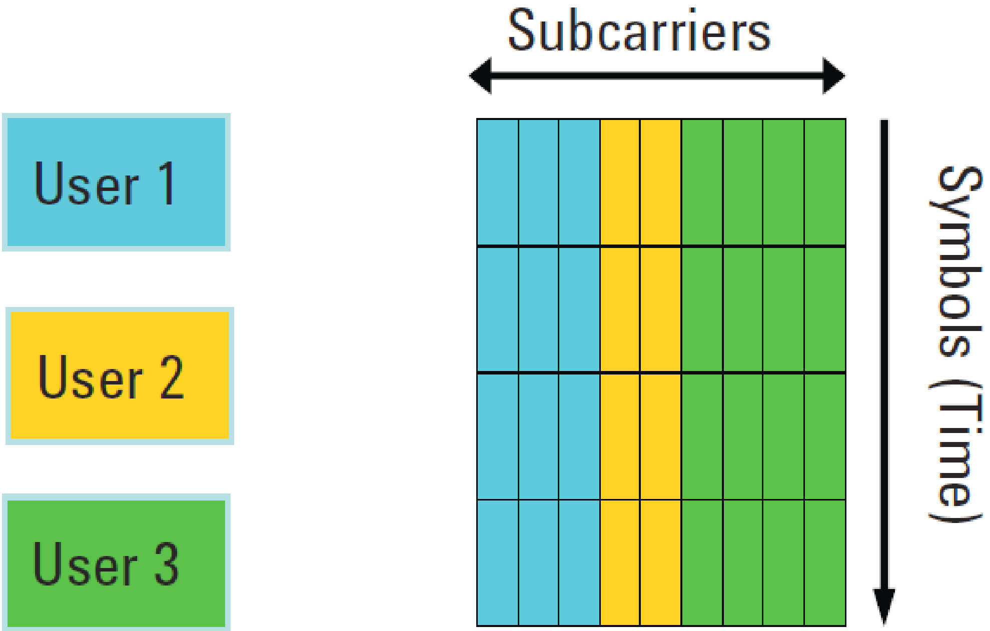


- A particular user is given all the subcarriers of the system for any specific OFDM symbol duration.
- Thus, the users are separated via time slots.
- All symbols allocated to all users are combined to form a OFDM-TDMA frame.
- Allows MS to reduce its power consumption, as the MS shall process only OFDM symbols which are dedicated to it.
- Different OFDM symbols can be allocated to different users based on certain allocation conditions.
- Since the OFDM-TDMA concept allocates the whole bandwidth to a single user, a reaction to different subcarrier attenuations could consist of leaving out highly distorted subcarriers

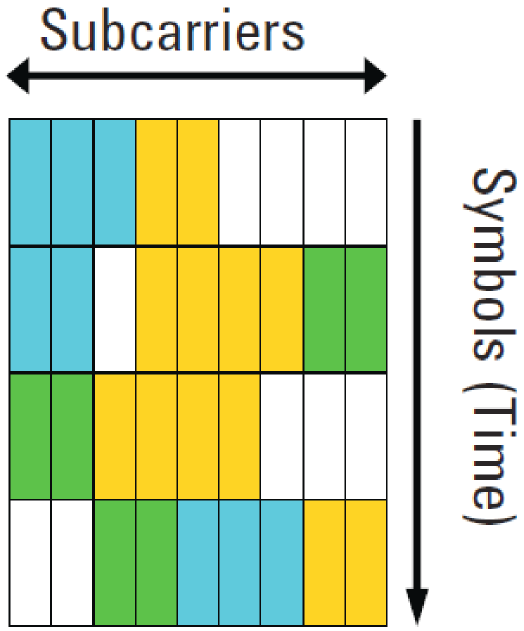
OFDMA

- Available subcarriers are distributed among all the users for transmission at any time instant.
- The subcarrier assignment is made at least for a time frame.
- Based on the subchannel condition, different baseband modulation schemes can be used for the individual subchannels
- The fact that each user experiences a different radio channel can be exploited by allocating only “good” subcarriers with high SNR to each user.
- The number of subchannels for a specific user can be varied, according to the required data rate.

OFDM-TDMA vs. OFDMA

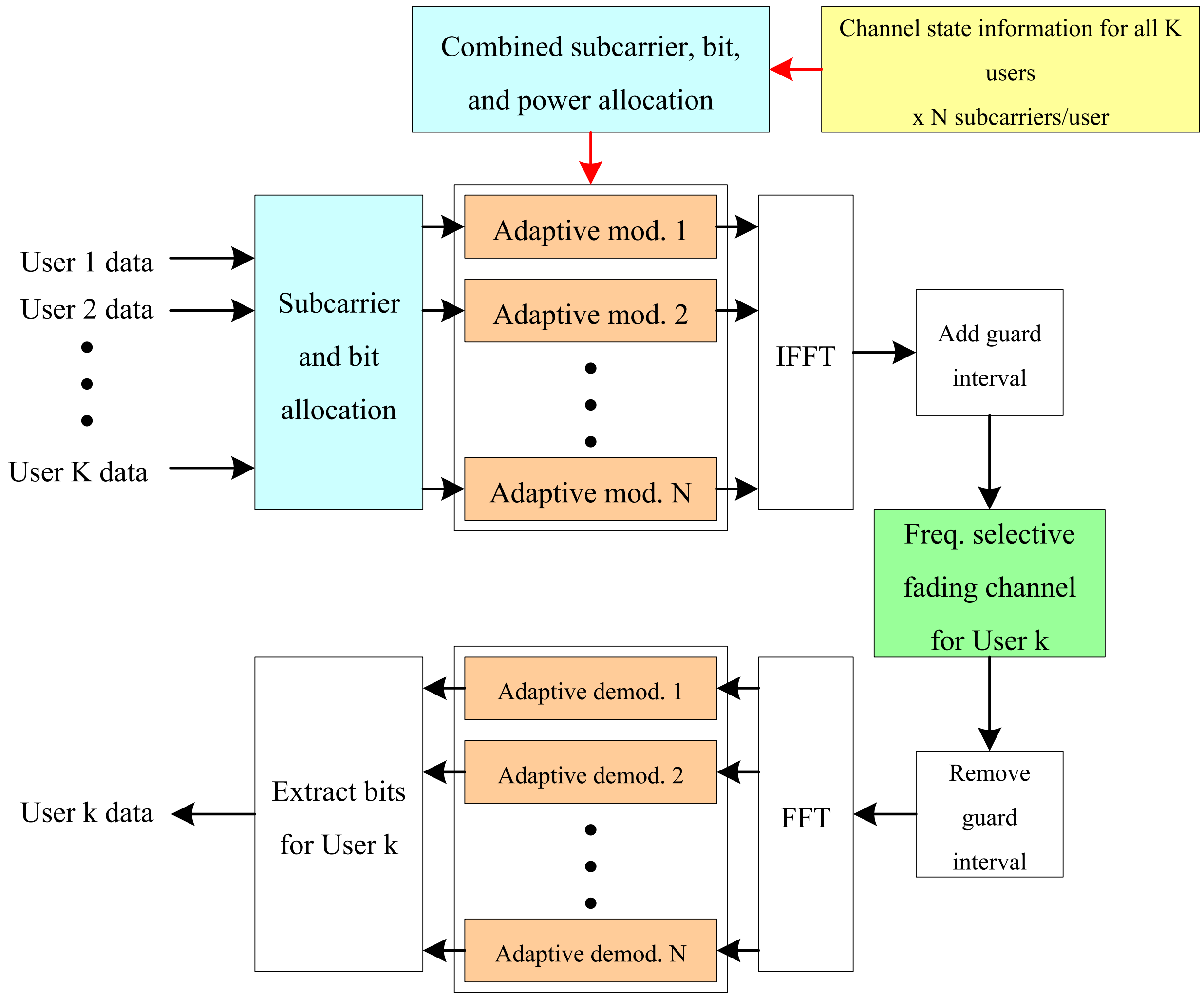


OFDM-TDMA



OFDMA

OFDMA Block Diagram



OFDM-CDMA

- User data are spread over several subcarriers and/or OFDM symbols using spreading codes, and combined with signal from other users.
- Several users transmit over the same subcarrier. In essence this implies **frequency-domain spreading**, rather than time-domain spreading, as it is conceived in a DS-CDMA system.